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Weak lighting functions and strong 26-surfaces ☆

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Abstract

The goal of this paper is to introduce the notion of weak lighting function in order to replicate the "continuous perception" associated with strong 26-surfaces. As a consequence, the continuous analogue defined ad hoc by Malgouyres and Bertrand only for these surfaces is extended for arbitrary objects, and the local characterization of finite strong 26-surfaces given in (Malgouyres and Bertrand, Int. J. Pattern Recognition Art. Intell. 13(4) (1999) 465–484) is generalized to possibly infinite surfaces. Moreover, weak lighting functions also replicate the "continuous perception" associated with (α, β)-surfaces, $(\alpha, \beta) \neq (6, 6)$, since they are generalizing the lighting functions previously defined by the authors.

Keywords: Continuous analogue; Lighting function; Digital manifold; Strong 26-surface; Digital connectivity; Digital Jordan–Brouwer theorem

1. Introduction

In a series of papers [1–3] we introduced an approach to the notion of digital space within a new framework for digital topology. This framework is presented as a multilevel architecture which provides a link between a device model, where the discrete nature of digital objects is represented, and an Euclidean space. The most elaborate notion of digital space was given in [3] by introducing the notion of lighting function, which intends to formalize the idea of "continuous perception" that an observer may take on digital objects. In this way, a digital space is not only determined by a device model but also by a lighting function defined on it. In some sense, such a

☆ This paper is an extended version with complete proofs of [4].

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function provides a method to construct a continuous analogue (an Euclidean polyhedron in fact) for each digital object, which is actually the "continuous perception" we are considering on that object.

Based on the idea of "face membership rule" from Kovalevsky [14], lighting functions were originally defined through a set of four axioms, yielding most of the "continuous perceptions" used in literature. In fact, for all $\alpha, \beta \in \{6, 18, 26\}$ there exist lighting functions providing the corresponding (α, β)-connectivity defined on $\mathbb{Z}^3$ within the graph-based approach to digital topology [11]. Moreover, (α, β)-surfaces [17, 9] are also found as surfaces in the corresponding digital spaces, for $(\alpha, \beta) \neq (6, 6)$. However these four axioms are not general enough to replicate the "continuous perception" associated with the generalization of $(26, 6)$-surfaces provided by the strong 26-surfaces [6, 15] (see Proposition 9.3).

The main goal of this paper is to introduce a set of axioms for lighting functions more general than that in [3, Definition 1]. These new axioms allow us to find a suitable digital space (R^3, f^{BM}) whose surfaces are exactly the set of strong 26-surfaces (Theorem 9.2). In addition, several consequences are derived from this result. Firstly, the continuous analogue defined by Malgouyres and Bertrand [15] only for strong 26-surfaces is now extended to arbitrary objects. Moreover, results proved in our framework with full generality, as the Digital Jordan–Brouwer Theorem, hold for strong 26-surfaces without a further proof. Finally, Theorem 9.2 provides an extension to possibly infinite strong 26-surfaces of the local characterization given in [15, Theorem 6] only for finite strong 26-surfaces.

These weak lighting functions are introduced in Section 3, where we also recall the definition of digital space and some other basic notions of our framework. In Section 4 the connectedness of digital objects in a digital space is defined in terms of the device model, and then characterized at all levels of our architecture. The main result in this section states that the connectedness of an object is characterized by the connectedness of its continuous analogue, and similarly for the complement of an object. This will enable us to use the continuous Jordan–Brouwer Theorem in order to prove easily the corresponding digital result (Theorem 5.3). Before, also in Section 5, a general notion of digital manifold is introduced by means of the continuous analogue; and we show, in Section 8, that the local notion of near strong 26-surface [15] provides a characterization for digital 2-manifolds (surfaces) in the digital space (R^3, f^{BM}) introduced in Section 7. The proof of this fact uses in a crucial way the Digital Jordan–Brouwer Theorem. This characterization is the main ingredient to show in Section 9 that the digital surfaces in (R^3, f^{BM}) are exactly the strong 26-surfaces in the sense of Bertrand and Malgouyres. In Section 6 we recall the notion of (near) strong 26-surface and some of its properties are restated appropriately in the language of our framework.

To ease the reading we collect in Section 2 the basic notions from polyhedral topology which are used through all the paper, while Appendix A contains other advanced notions and results which are needed in the proofs of Section 8. We refer to [19, 21] for further details on polyhedral topology.

2. Basic notions and notations

A *polytope* σ is the convex hull $\langle x_0, x_1, \ldots, x_m \rangle$ of a finite set of points $A = \{x_i\}_{i=0}^{m}$ in some Euclidean space $\mathbb{R}^d$. The *dimension of* σ, $\dim \sigma$, is the dimension of the affine variety spanned by A; and so, σ is called an *n-dimensional polytope* (or, simply, an *n-polytope*) if $\dim \sigma = n$. Notice that, for all $n \geqslant 1$, every n-polytope is homeomorphic to the Euclidean n-ball B^n. So, the notions of *interior* of σ, $\mathring{\sigma}$, and *boundary* of σ, $\partial\sigma$, are clear. As usual for a 0-polytope σ (i.e., a polytope consisting of a single point) $\mathring{\sigma} = \sigma$ and $\partial\sigma = \emptyset$.

Given a polytope σ, any subset $\tau \subseteq \partial\sigma$ is called a *proper face* of σ if there exists an hyperplane H^{d-1} such that $H^{d-1} \cap \sigma = H^{d-1} \cap \partial\sigma = \tau$, while the empty set and σ itself are called *improper faces*. We write $\tau \leqslant \sigma$ if τ is a face of σ, and $\tau < \sigma$ if it is a proper face. Notice that each proper face $\tau < \sigma$ is a polytope itself; moreover, $\partial\sigma = \bigcup\{\tau \mid \tau < \sigma\}$ and $\mathring{\sigma} = \sigma - \partial\sigma$. The 0-dimensional faces of σ are usually called the *vertices* of σ. A *simplex* is a polytope whose vertices are affinely independent points.

A *polyhedral complex* is a set K of polytopes in some Euclidean space $\mathbb{R}^d$ satisfying:
1. If $\sigma \in K$ and $\tau < \sigma$ then $\tau \in K$.
2. If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is a face of both σ and τ.

A *simplicial complex* is a polyhedral complex whose cells are simplexes. Actually, polyhedral complexes are particular cases of cellular complexes, as they are usually defined in polyhedral topology. So, for simplicity, and where there is no place to confusion, we will usually call a *complex* to any polyhedral complex K, and the polytopes of K will be simply called *cells* in next sections.

The *underlying polyhedron* of a complex K is the space $|K| = \bigcup\{\sigma \mid \sigma \in K\}$ endowed with the weak topology defined by the polytopes of K; namely, $C \subseteq |K|$ is a closed subset if and only if $C \cap \sigma$ is a closed subset for each $\sigma \in K$. If K is a simplicial complex, then K is called a *triangulation* of $|K|$.

A complex K is said to be *locally finite* if for each $x \in |K|$ there exists a neighbourhood of x which intersects only a finite number of cells in K. In particular, the set $\{\tau \in K \mid \sigma < \tau\}$ is finite for each $\sigma \in K$ whenever K is a locally finite complex. It is well-known that if K is a locally finite complex in the Euclidean space $\mathbb{R}^d$, then the weak topology on $|K|$ coincides with the topology of $|K|$ as a subspace of $\mathbb{R}^d$.

The *dimension of* K, $\dim K$, is the largest dimension of its cells. We say that K is *n-dimensional* if $\dim K = n$, and *homogeneously n-dimensional* if every cell in K is a face of some n-cell $\sigma \in K$.

A *subcomplex of* K is a complex L such that $L \subseteq K$. Notice that for each polytope $\sigma \in K$ both σ and $\partial\sigma$ naturally define subcomplexes of K. The *r-skeleton of* K is the subcomplex $\mathrm{sk}^r(K) = \{\sigma \in K \mid \dim \sigma \leqslant r\}$. If $\sigma \in K$, the *star* and the *link* of σ are respectively the subcomplexes $\mathrm{st}(\sigma; K) = \{\tau \in K \mid \tau \leqslant \delta,\ \sigma \leqslant \delta \text{ and } \delta \in K\}$ and $\mathrm{lk}(\sigma; K) = \{\tau \in \mathrm{st}(\sigma; K) \mid \tau \cap \sigma = \emptyset\}$. If K is simplicial, $L \subseteq K$ is called a *full subcomplex* if any $\sigma \in K$ whose vertices are in L belongs to L.

Given a complex K, we say that the complex L is a subdivision of K if $|K|=|L|$ and each polytope of L is contained in some polytope of K. We shall now describe the construction of a subdivision which plays an important role in polyhedral topology. A *centroid-map* is a map $c : K \to |K|$ such that $c(\sigma) \in \mathring{\sigma}$. The point $c(\sigma)$ is called the *centroid* of σ. The *derived subdivision of* K induced by the centroid-map c is the simplicial complex $K^{(1)}$ whose vertices are the centroids of the polytopes of K and whose simplexes are of the form $\langle c(\sigma_0), c(\sigma_1), \ldots, c(\sigma_m)\rangle$, where $\sigma_0 < \sigma_1 < \cdots < \sigma_m$.

A map $f : P \to Q$ between polyhedra is *piecewise linear* (abbreviated *pl-map*) if there are triangulations $P=|K|$ and $Q=|L|$ for which f maps vertices to vertices and it is linear on each simplex of K. A bijective pl-map whose inverse is also a pl-map is called a *pl-homeomorphism*. A *polyhedral n-ball* (*n-sphere*) is a polyhedron which is pl-homeomorphic to an n-simplex (the boundary of an $(n+1)$-simplex, respectively). A simplicial complex K is called a *combinatorial n-manifold* if the link of every k-simplex is either a polyhedral $(n-k-1)$-sphere or $(n-k-1)$-ball. It is known that, if K is a combinatorial manifold, any other triangulation of the polyhedron $|K|$ is also a combinatorial manifold, and so a polyhedron M is called a *polyhedral manifold* (*pl-manifold*) if it can be triangulated as a combinatorial manifold. The *boundary* of K is the subcomplex $\partial K = \{\sigma \in K \mid \mathrm{lk}(\sigma; K) \text{ is a } pl\text{-ball}\}$. When $\partial K = \emptyset$ we say that K is a combinatorial manifold without boundary. Since a pl-homeomorphism is a topological homeomorphism, any point of an n-dimensional pl-manifold $|K|$ admits a neighbourhood which is topologically homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}$. Hence any n-dimensional pl-manifold $|K|$ is a (triangulated) topological manifold whose boundary is $|\partial K|$.

We conclude this section by giving some notations and definitions from graph theory.

A *directed graph* on a given set X is a couple $G(X,E)$, where $E \subseteq X \times X$ is a binary relation on X. The elements of X and E are respectively called vertices and edges of G. A *directed path* in G is a sequence $\Gamma = (x_i)_{i=0}^n$ of vertices such that $(x_{i-1}, x_i) \in E$ $(1 \leqslant i \leqslant n)$; the number n is the *length* of Γ. The path Γ is a *cycle* if $x_0 = x_n$. A directed graph without cycles is said to be *acyclic*, and it is called *transitive* if it contains an edge (x,y) whenever there is a directed path from x to y. A path Γ is said to be *undirected* if either (x_{i-1}, x_i) or (x_i, x_{i-1}) belongs to E.

A *subgraph* of $G(X,E)$ is a graph $G'(X',E')$ such that $X' \subseteq X$ and $E' \subseteq E$. If, in addition, $E' = E \cap (X' \times X')$, G' is said to be a *full subgraph* or the subgraph of G *induced* by X'. The connection *via* undirected paths in a graph $G(X,E)$ defines an equivalence relation on the set X. The *components* of G are the subgraphs induced by the equivalence classes of X under this relation, and G is said to be *connected* if it has only one component.

A graph $G(X,E)$ is said to be *undirected* if E is symmetric (i.e., $(y,x) \in E$ whenever $(x,y) \in E$), and then it is considered that there is only one edge between the vertices x and y written $\{x,y\}$.

Finally, we write $\mathscr{P}(X)$ to denote the family of all subsets of a given set X.

3. Digital spaces

In [3] we introduced a notion of digital space that describes the discrete structure of digital images and provides a continuous interpretation for them as well. In this setting the continuous interpretation of a digital image is represented by a polyhedron called its continuous analogue. The goal of this section is to generalize our notion of digital space in order to deal with a wider family of continuous analogues for digital images.

The spatial layout of pixels in digital images is represented by a *device model* which is a homogeneously n-dimensional locally finite polyhedral complex K. Only the n-cells in a device model K are representing pixels, while the other lower dimensional cells in K are used to describe how the pixels could be linked to each other. In this way, the objects displayed in digital images are subsets of the set $\mathrm{cell}_n(K)$ of n-cells in a device model K; and, thus, we call a *digital object in* K to any subset $O \subseteq \mathrm{cell}_n(K)$.

Obviously, all complexes that fulfil the above conditions are examples of device models. However, it is worth to point out that the graph-based approach to digital topology provides another source of examples for this notion. In this approach, a number of different grids and adjacency relations have been considered [11], most of them based on the Voronoi neighbourhoods of the grid points. Given a grid P of points in a Euclidean space $\mathbb{R}^n$, the Voronoi neighbourhood of $p \in P$ is the locus of all points in $\mathbb{R}^n$ closer to p than to any other grid point. A Voronoi neighbourhood is always a convex set. Furthermore, if there exists a constant $D > 0$ such that any point of $\mathbb{R}^n$ is within a distance less than D of some grid point, then the Voronoi neighbourhoods are also compact polytopes. In this sense, these grids can be understood as device models.

In particular, in this paper we will mainly deal with the device model R^n associated with the grid $\mathbb{Z}^n \subseteq \mathbb{R}^n$ of all points with integer coordinates. The Voronoi neighbourhood of a grid point $p \in \mathbb{Z}^n$ is the unit n-cube centered at p whose edges are parallel to the coordinate axes. So that, every digital object O in R^n, which is a subset of these n-cubes, can be identified with a subset of points in $\mathbb{Z}^n$. Henceforth we shall use this identification without further comment. This device model R^n is called the *standard cubical decomposition of the Euclidean n-space*.

Another interesting device model is the tiling of the Euclidean plane by regular hexagons. In this model each hexagon is the Voronoi neighbourhood of its centre.

Before proceeding with the definition of digital space, we need some notions, which are illustrated in Fig. 1 for an object O in the device model R^2.

The first two notions formalize two types of "digital neighbourhoods" of a cell $\alpha \in K$ in a given digital object $O \subseteq \mathrm{cell}_n(K)$. Indeed, we call the *star of* α *in* O to the set $\mathrm{st}_n(\alpha; O) = \{\sigma \in O \mid \alpha \leqslant \sigma\}$ of n-cells (pixels) in O having α as a face. Similarly, the *extended star of* α *in* O is the set $\mathrm{st}_n^*(\alpha; O) = \{\sigma \in O \mid \alpha \cap \sigma \neq \emptyset\}$ of n-cells (pixels) in O intersecting α.

The third notion is the *support* of a digital object O which is defined as the set $\mathrm{supp}(O)$ of cells of K (not necessarily pixels) that are the intersection of n-cells (pixels) in O. Namely, $\alpha \in \mathrm{supp}(O)$ if and only if $\alpha = \bigcap\{\sigma \mid \sigma \in \mathrm{st}_n(\alpha; O)\}$.

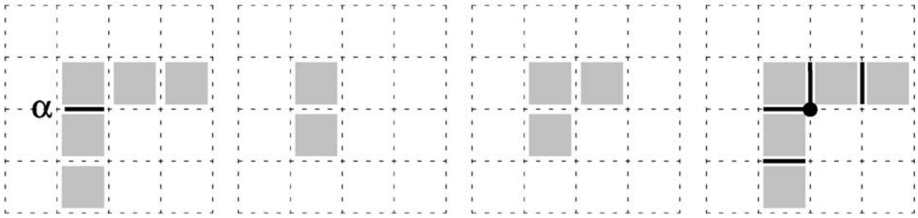


Fig. 1. The physical support of an object O and two types of digital neighbourhoods in O for a cell α. The cells in O together with the bold edges and dots are the elements in $\mathrm{supp}(O)$.

To ease the writing, when the digital object is the whole set $\mathrm{cell}_n(K)$ we shall simply write $\mathrm{supp}(K)$, $\mathrm{st}_n(\alpha; K)$ and $\mathrm{st}_n^*(\alpha; K)$ instead of $\mathrm{supp}(\mathrm{cell}_n(K))$, $\mathrm{st}_n(\alpha; \mathrm{cell}_n(K))$ and $\mathrm{st}_n^*(\alpha; \mathrm{cell}_n(K))$, respectively.

Next lemma is immediate from these definitions.

Lemma 3.1. *Let K be a device model. The following properties hold for any digital object O in K and any cell $\alpha \in K$:*

1. *If $\dim \alpha = 0$ then $st_n(\alpha; O) = st_n^*(\alpha; O)$.*
2. *Let $\{v_i\}_{i=0}^k$ be the set of vertices of α. Then $st_n(\alpha; O) = \bigcap_{i=0}^k st_n(v_i; O)$ and $st_n^*(\alpha; O) = \bigcup_{i=0}^k st_n(v_i; O)$.*
3. *$st_n(\alpha; O) = st_n(\alpha; st_n(\alpha; O)) = st_n(\alpha; st_n^*(\alpha; O))$.*
4. *$st_n^*(\alpha; O) = st_n^*(\alpha; st_n^*(\alpha; O))$.*
5. *If $\alpha \in O$ then $\alpha \in \mathrm{supp}(O)$.*
6. *$\alpha \in \mathrm{supp}(O)$ is equivalent to $\alpha \in \mathrm{supp}(st_n(\alpha; O))$ or alternatively to $\alpha \in \mathrm{supp}(st_n^*(\alpha; O))$.*

In some sense, the support of a digital object is the minimum set of cells representing the physical layout of that object. However, one of the key ideas of our approach is to distinguish this physical representation from the "continuous perception" that an observer may take on the object. In fact, we admit possible different perceptions on the same physical support. The continuous perception of a digital object O is formalized through the notion of continuous analogue associated with O, which is defined below as a certain Euclidean polyhedron. But, in general, the continuous analogue is topologically distinguishable from the support. Actually, a given digital object whose support is a two-dimensional polyhedron may have a one-dimensional continuous analogue.

In [3], the continuous analogue of each digital object in a device model K is obtained from a lighting function defined on K. In order to reach our goal, we next generalize these functions as follows.

Definition 3.2. Given a device model K, a function $f : \mathscr{P}(\mathrm{cell}_n(K)) \times K \to \{0, 1\}$ is said to be a *weak lighting function* (w.l.f.) on K if it verifies the following five properties for all $O \subseteq \mathrm{cell}_n(K)$ and $\alpha \in K$.

1. If $\alpha \in O$ then $f(O,\alpha)=1$.
2. If $\alpha \notin \mathrm{supp}(O)$ then $f(O,\alpha)=0$.
3. $f(O,\alpha) \leqslant f(\mathrm{cell}_n(K),\alpha)$.
4. $f(O,\alpha)=f(\mathrm{st}_n^*(\alpha;O),\alpha)$.
5. Let $O_1 \subseteq O_2 \subseteq \mathrm{cell}_n(K)$ and $\alpha \in K$ such that $\mathrm{st}_n(\alpha;O_1)=\mathrm{st}_n(\alpha;O_2)$, $f(O_1,\alpha)=0$ and $f(O_2,\alpha)=1$. Then, the set of cells $\alpha(O_1;O_2)=\{\beta<\alpha \mid f(O_1,\beta)=0,\ f(O_2,\beta)=1\}$ is non-empty and connected in $\partial\alpha$. Moreover, if O is a digital object containing O_2, then $f(O,\beta)=1$ for every $\beta \in \alpha(O_1;O_2)$.

If $f(O,\alpha)=1$ we say that the w.l.f. f *lights* the cell α for the digital object O.

A *digital space* is then defined as a pair (K,f) where K is a device model and f is a weak lighting function on K.

The ideas underlying properties (1)–(5) in the previous definition are quite intuitive. We will postpone their explanation to the end of this section since they are strongly related to the notion of continuous analogue.

Example 3.3. Every device model $K \neq \emptyset$ admits the weak lighting functions $f_{\max}$, $f_{\min}$ and g given, respectively, by

1. $f_{\max}(O,\alpha)=1$ if and only if $\alpha \in \mathrm{supp}(O)$
2. $f_{\min}(O,\alpha)=1$ if and only if $\alpha \in O$
3. $g(O,\alpha)=1$ if and only if $\alpha \in \mathrm{supp}(O)$ and $\mathrm{st}_n(\alpha;K) \subseteq O$

Both $f_{\max}$ and g are distinct from $f_{\min}$ only if there exist two n-cells in K with a common face. On the other hand, if a cell $\alpha \in K$ is the intersection of a proper subset of n-cells in $\mathrm{st}_n(\alpha;K)$ then $g \neq f_{\max}$. Notice that this is not the case in the hexagonal tiling of the Euclidean plane; so that, the weak lighting functions $f_{\max}$ and g coincide on this device model.

Observe that the family of all w.l.f.'s on a given device model K is a partially ordered set by defining $f' \leqslant f$ if and only if $f'(O,\alpha) \leqslant f(O,\alpha)$, for any object $O \subseteq \mathrm{cell}_n(K)$ and any cell $\alpha \in K$. It is easy to show that $f_{\max}$ and $f_{\min}$ are, respectively, the greatest and the least elements for this ordering.

In [3, Definition 1] we defined the notion of a lighting function. In that definition properties (1), (2) and (3) already appeared as (F2), (F1) and (F4), respectively. However property

(F3) $f(O,\alpha)=f(\mathrm{st}_n(\alpha;O),\alpha)$

in [3] is here replaced by properties (4) and (5) above. From the equality $\mathrm{st}_n(\alpha;O)=\mathrm{st}_n(\alpha;\mathrm{st}_n^*(\alpha;O))$ it is readily checked that property (F3) implies property (4). Moreover, if (F3) holds then no cell in K satisfies all hypothesis in (5). Hence (F3) also implies property (5). So that, w.l.f.'s generalize lighting functions in [3]. Notice that all the three functions in Example 3.3 satisfy property (F3). In order to show that the class of w.l.f.'s strictly contains all lighting functions we give the following

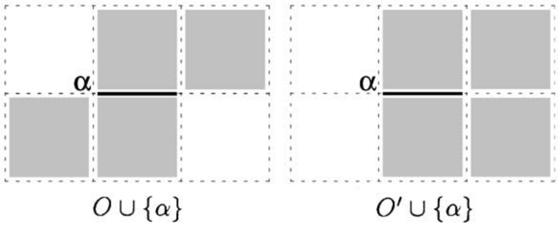


Fig. 2. The w.l.f. h in Example 3.4 does not satisfy property (F3).

Example 3.4. Let h be the w.l.f. defined on R^2 by $h(O,\alpha)=1$ if and only if: (a) $\dim\alpha=2$ and $\alpha\in O$; (b) $\dim\alpha=0$ and $\mathrm{st}_2(\alpha;R^2)\subseteq O$; (c) $\dim\alpha=1$ and one of the two following conditions holds:
(c1) $\mathrm{st}_2^*(\alpha;O)=\mathrm{st}_2^*(\alpha;R^2)$
(c2) $\alpha\in\mathrm{supp}(O)$ and there exist $\sigma,\tau\in\mathrm{st}_2^*(\alpha;R^2)-O$ such that $\sigma\cap\tau=\emptyset$.

To check that h does not satisfy property (F3), one observes that for the digital objects O and O' and the 1-cell α in Fig. 2 the equality $\mathrm{st}_2(\alpha;O)=\mathrm{st}_2(\alpha;O')$ holds; however the definition of h yields that $h(O,\alpha)=1$ while $h(O',\alpha)=0$.

In order to associate a continuous analogue with each digital object, we consider some other intermediate levels. To introduce them we choose an arbitrary but fixed centroid-map $c:K\to|K|$ on the polyhedral complex K.

The *device level* of O is the pair $(K(O),f_O)$, where $K(O)=\{\alpha\in K\mid\alpha\leqslant\sigma,\ \sigma\in O\}$ is the subcomplex of K induced by the cells in O, and f_O is the function defined on $K(O)$ by $f_O(O',\alpha)=f(O,\alpha)f(O',\alpha)$, for $O'\subseteq O$ and $\alpha\in K(O)$. It is not difficult to show that f_O is actually a w.l.f. and so $(K(O),f_O)$ is a digital space. Notice, however, that the plain restriction of f to the set $\mathscr{P}(O)\times K(O)$ does not satisfy property (3) in Definition 3.2.

The *logical level* of O is an undirected graph, $\mathscr{L}_O^f$, whose vertices are the centroids of n-cells in O and two of them $c(\sigma),c(\tau)$ are adjacent if there exists a common face $\alpha\leqslant\sigma\cap\tau$ such that $f(O,\alpha)=1$.

The *conceptual level of O* is the directed graph $\mathscr{C}_O^f$ whose vertices are the centroids $c(\alpha)$ of all cells $\alpha\in K$ with $f(O,\alpha)=1$, and its directed edges are $(c(\alpha),c(\beta))$ with $\alpha<\beta$.

The *simplicial analogue* of O is the order complex $\mathscr{A}_O^f$ associated to the directed graph $\mathscr{C}_O^f$. That is, $\langle c(\sigma_0),c(\sigma_1),\dots,c(\sigma_m)\rangle$ is an m-simplex of $\mathscr{A}_O^f$ if the sequence $c(\sigma_0),c(\sigma_1),\dots,c(\sigma_m)$ is a directed path in $\mathscr{C}_O^f$; or, equivalently, $\sigma_0<\sigma_1<\cdots<\sigma_m$ are cells in K lighted for the digital object O.

Notice that, for any object O, the conceptual level $\mathscr{C}_O^f$ is an acyclic, transitive and locally finite directed graph. From these properties it is easily checked that the simplicial analogue $\mathscr{A}_O^f$ is in fact a simplicial complex; moreover, it is a full subcomplex of the derived subdivision $K^{(1)}$ induced in K by the centroid-map c.

This leads us to define the *continuous analogue of O* as the underlying polyhedron $|\mathscr{A}_O^f|$ of the simplicial complex $\mathscr{A}_O^f$.

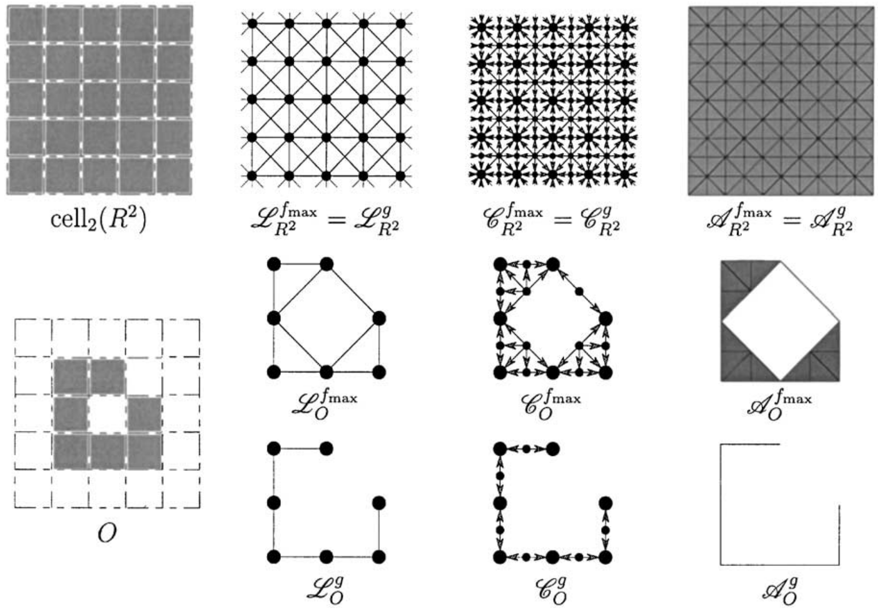


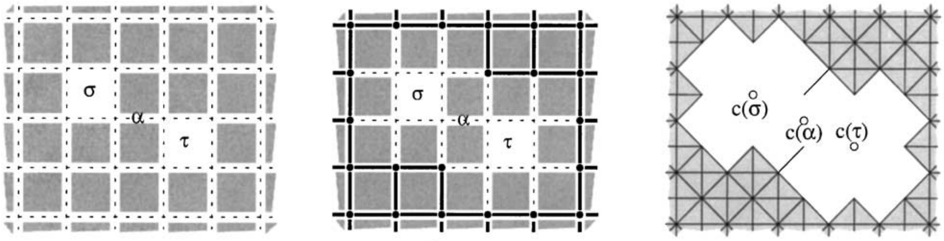
Fig. 3. Levels of two digital objects, O and $\mathrm{cell}_2(R^2)$, for the w.l.f.'s $f_{\max}$ and g in Example 3.3.

In Fig. 3 are shown a digital object O and its levels for two distinct lighting functions. This illustrates how the continuous analogue of an object directly depends on the considered lighting function. However, a digital object may have the same levels for two different lighting functions. This is the case, for the w.l.f.'s $f_{\max}$ and g, of the object $\mathrm{cell}_2(R^2)$, consisting of all the pixels of the device model R^2. Notice, also in Fig. 3, that the support of the object O is a two-dimensional polyhedron, while its continuous analogue $|\mathscr{A}^g_O|$ is one-dimensional.

For the sake of simplicity, we will usually drop "f" from the notation of the levels of an object. Moreover, for the whole object $\mathrm{cell}_n(K)$ we will simply write $\mathscr{L}_K$, $\mathscr{C}_K$ and $\mathscr{A}_K$ for its levels.

It is worth to point out that several other authors have already used polyhedra to define a "continuous analogue" of digital objects. Some of these continuous analogues are only defined for a particular family of objects (for example, surfaces in [9, 15]), while other definitions stand for arbitrary objects in the grid $\mathbb{Z}^3$ [20, 13, 10, 12]. Most of the continuous analogues in the literature can be found as particular cases of our construction for suitable digital spaces (see [1, 3] and Theorem 8.2 in this paper). Indeed, our notion of continuous analogue is more general, since it works for any grid of points.

In any case, continuous analogues are used to take advantage of the powerful machinery of polyhedral topology in order to prove results in digital topology as well as to check that new digital notions represent accurately the usual ones defined over the continuous analogue of objects; this will be the case of our notion of digital connectedness (see Section 4). However it is also possible to proceed along the inverse way. That is, we may directly say that an object satisfies the digital counterpart



(a) O_1 (b) $\{\gamma \in R^2 \mid f(O_1, \gamma) = 1\}$ (c) $|\mathscr{A}_{O_1}| \cup \{c(\alpha), c(\sigma), c(\tau)\}$

Fig. 4. Property (5) in Definition 3.2 is required to obtain a right representation of the connectivity of the complement of any object.

of a given topological property P if its continuous analogue satisfies P. Then it is needed to characterize digitally that property, or at least to find sufficient conditions, at a level as close to the logical one as possible, to ensure that an object satisfies it; this will be the case of our notion of digital manifold (see Section 5).

We finish this section explaining the intuitive ideas underlying properties (1)–(5) in the definition of weak lighting function (Definition 3.2).

Property (1) expresses that the n-cells in a digital object O are always perceived when we look at this object. In addition to these pixels, we can only perceive cells in the physical support of O, $\mathrm{supp}(O)$, by property (2). Actually, the lighting function determines what lower dimensional cells in $\mathrm{supp}(O)$ are perceived.

Property (3) is needed to ensure that the continuous analogue $|\mathscr{A}_O|$ of any object O is a subspace of the continuous analogue $|\mathscr{A}_K|$ of the whole space $\mathrm{cell}_n(K)$. Moreover, it is straightforwardly checked from property (3) that the logical level $\mathscr{L}_O$ is a (non necessarily full) subgraph of $\mathscr{L}_K$, $\mathscr{C}_O$ is a full subgraph of $\mathscr{C}_K$, and $\mathscr{A}_O$ is a full subcomplex of $\mathscr{A}_K$.

Finally, both properties (F3) in [3] and (4) in this paper state that whether a cell α is lighted or not for a given object O is a local property of the object, and so it depends on the pixels of O in certain vicinity of α. In property (F3) we choose the smallest "digital neighbourhood" of α in O, namely $\mathrm{st}_n(\alpha; O) = \mathrm{st}(\alpha; K) \cap O$, to state this local condition. However, $|\mathrm{st}(\alpha; O)|$ is not, in general, a neighbourhood of the cell α in the polyhedron $|K|$. Due to this, we enlarge the vicinity of α in O up to $\mathrm{st}_n^*(\alpha; O) = N(\alpha; K) \cap O$, where $N(\alpha; K) = \{\beta \in K \mid \beta \leqslant \gamma \in K \text{ and } \gamma \cap \alpha \neq \emptyset\}$ is the smallest subcomplex of K which is also a neighbourhood of α in $|K|$. This new local condition provides a strictly more general family of lighting functions, as shown in Example 3.4. But this family must be restricted since, otherwise, our continuous analogue would not provide a right interpretation of the connectivity of complements of objects, as we show in Example 3.5 below. To avoid this, we require in addition property (5); see Theorem 4.2.

Example 3.5. On the device model R^2, we consider the function f given by $f(O, \alpha) = 1$ if and only if $\alpha \in O$ or $\mathrm{st}_2^*(\alpha; R^2) \subseteq O$. Notice that f satisfies properties (1)–(4) in Definition 3.2. However property (5) fails for the 1-cell α and the objects O_1 and $\mathrm{cell}_2(R^2)$ in Fig. 4(a). To check the latter one observes that $f(\mathrm{cell}_2(R^2), \beta) = 1$ for every cell $\beta \in R^2$, while $f(O_1,\beta)=1$ if and only if $\beta \in O_1$ or $\sigma \cap \beta = \emptyset = \tau \cap \beta$ (see Fig. 4(b)). Moreover, this defines the continuous analogue of O_1 as the polyhedron $|\mathscr{A}_{O_1}|$ shown in grey colour in Fig. 4(c), and $|\mathscr{A}_{R^2}| = \mathbb{R}^2$. Hence, the complement $|\mathscr{A}_{R^2}| - |\mathscr{A}_{O_1}|$ of the continuous analogue of O_1 (the white hole in Fig. 4(c) containing $c(\sigma)$ and $c(\tau)$) is a connected space, which does not correspond with our intuitive perception of the complement $\mathrm{cell}_2(R^2) - O_1$ of O_1 consisting of two isolated pixels σ, τ.

4. Connectivity in digital spaces

It is well known that, in order to avoid connectivity paradoxes, two different notions of connection, one for digital objects and the other for their complements, must be defined in most of the grids considered in the graph-theoretical approach to digital topology. Both notions can be found as particular cases of our more general definition of connection for pairs of digital objects, which was introduced in [3] in terms of the device model of a digital space as follows.

Definition 4.1. Let O and O' be two disjoint digital objects in a digital space (K, f). Two n-cells $\sigma, \tau \in O$ are said to be *O'-adjacent in O* if there exists a common face $\alpha \leqslant \sigma \cap \tau$ such that $f(O',\alpha)=0$ and $f(O \cup O',\alpha)=1$. An *O'-path in O from σ to τ* is a finite sequence $(\sigma_i)_{i=0}^m \subseteq O$ such that $\sigma_0=\sigma$, $\sigma_m=\tau$ and σ_{i-1} is O'-adjacent in O to σ_i, for $i=1,\dots,m$.

Notice that each $\sigma \in O$ is O'-adjacent to itself, since $f(O \cup O',\sigma)=1$ and $f(O',\sigma)=0$. Thus the existence of O'-paths defines an equivalence relation on O. We call each equivalence class of O under this relation an *O'-component of O*; and we say that O is *O'-connected* if it has only one O'-component. It is not evident from definitions that the O'-components of O are O'-connected digital objects themselves. We will prove this property in Remark 4.6 below.

Given a digital object O in a digital space (K, f) the previous definitions provide an entire family of notions of connection for O in relation to any object $O' \subseteq \mathrm{cell}_n(K) - O$. The extreme cases, when $O'=\emptyset$ and $O'=\mathrm{cell}_n(K)-O$, represent the connectivity of the digital object O itself and the connectivity of O as the complement of the object $O'=\mathrm{cell}_n(K)-O$, respectively. So, we will usually drop any reference to the object O' in the special case $O'=\emptyset$, and we will simply talk about adjacency, paths, components and the connectedness of the object O.

As it was remarked in Section 3, we must check the accuracy of Definition 4.1 showing that the connectivity of any digital object agrees with the connectivity of its continuous analogue. This will be done in the following theorem by characterizing the O'-components of an object O at each level of our architecture. Below, $L_1 \backslash L_2 = \{\alpha \in L_1 \mid \alpha \cap |L_2| = \emptyset\}$ will stand for the simplicial complement of L_2 in L_1, where L_1 and L_2 are subcomplexes of a simplicial complex L; and if G, H are graphs, $G \backslash H$ denotes the subgraph of G induced by the vertices which are not in H.

Theorem 4.2. *Let O and O' be two disjoint digital objects in a digital space. The family $\mathscr{F}$ of O'-components of O can be described in any of the following ways*
1. *Conceptual level*: $\mathscr{F}=\{O_G\}$, *where* $O_G=\{\sigma\in O\,|\,c(\sigma)$ *is a vertex of* $G\}$, *and* G *ranges over the family of components of the directed graph* $\mathscr{C}_{O\cup O'}\backslash\mathscr{C}_{O'}$.
2. *Simplicial level*: $\mathscr{F}=\{O_A\}$, *where* $O_A=\{\sigma\in O\,|\,c(\sigma)\in A\}$, *and* A *ranges over the family of components of the simplicial complement* $\mathscr{A}_{O\cup O'}\backslash\mathscr{A}_{O'}$.
3. *Continuous level*: $\mathscr{F}=\{O_X\}$, *where* $O_X=\{\sigma\in O\,|\,c(\sigma)\in X\}$, *and* X *ranges over the family of components of the space* $|\mathscr{A}_{O\cup O'}|-|\mathscr{A}_{O'}|$.

For lighting functions in [3] this result was already stated and its proof sketched. However, the proof of Theorem 4.2 for arbitrary weak lighting functions as introduced in this paper is quite a lot elaborate, and we consider interesting to include it here in detail. The first goal will be to prove part (1) directly from definitions (see Theorem 4.10). Then, part (2) is an immediate consequence of part (1), since the directed graph $\mathscr{C}_{O\cup O'}\backslash\mathscr{C}_{O'}$ is the 1-skeleton of the simplicial complement $\mathscr{A}_{O\cup O'}\backslash\mathscr{A}_{O'}$. Finally, to derive part (3) from part (2) we will require some additional lemmas from polyhedral topology; see Theorem 4.14.

Before proceeding with the proof of Theorem 4.2, observe that, in general, O'-connectedness cannot be characterized at the logical level of our architecture. For example, the complement $\mathrm{cell}_2(R^2)-O$ of the object O shown in Fig. 3 is not O-connected in the digital space $(R^2,f_{\max})$, while the complement $\mathscr{L}_{R^2}^{f_{\max}}\backslash\mathscr{L}_{O}^{f_{\max}}$ of its logical level is a connected graph. Despite this, the connectedness of digital objects can be characterized at all levels of our architecture as stated in the following.

Theorem 4.3. *For a given digital object O, the following properties are equivalent*: (1) *O is connected* (*i.e.*, *$\emptyset$-connected*); (2) *$\mathscr{L}_O$ is a connected graph*; (3) *$\mathscr{C}_O$ is a connected directed graph*; (4) *$\mathscr{A}_O$ is a connected complex*; *and* (5) *$|\mathscr{A}_O|$ is a connected polyhedron.*

Proof. It is immediate from definitions that (1) and (2) are equivalent, while the rest of equivalences are consequences of Theorem 4.2. □

To prove part (1) in Theorem 4.2 we will need the following technical lemmas. The first one is an immediate consequence of property (4) in the definition of w.l.f.'s and the equality $\mathrm{st}_n(\alpha;O)=\mathrm{st}_n^*(\alpha;O)$ for α any vertex in K (see Lemma 3.1(1)).

Lemma 4.4. *Let $O_1\subseteq O_2$ be two objects in a digital space (K,f). If $\alpha\in K$ is a vertex with $st_n(\alpha;O_1)=st_n(\alpha;O_2)$ then $f(O_1,\alpha)=f(O_2,\alpha)$.*

Lemma 4.5. *Let $O_1\subseteq O_2$ be two objects in a digital space (K,f) and $\alpha\in K$ such that $f(O_1,\alpha)=0$, $f(O_2,\alpha)=1$ and $st_n(\alpha;O_1)=st_n(\alpha;O_2)$. Then, there exists $\beta\in\alpha(O_1;O_2)=\{\gamma<\alpha\,|\,f(O_1,\gamma)=0,\ f(O_2,\gamma)=1\}$ such that $st_n(\beta;O_1)$ is a proper subset of $st_n(\beta;O_2)$.*

Proof. The result will be proved by induction on the dimension of α.

Firstly, notice that $\dim\alpha > 0$ by Lemma 4.4; so, assume $\dim\alpha = 1$. Property (5) of w.l.f.'s guarantees that there exists a proper face β of α in the set $\alpha(O_1;O_2)$. Then, the result holds for β by Lemma 4.4 again since $\dim\beta = 0$.

If $\dim\alpha \neq 1$, let $\gamma \in \alpha(O_1;O_2)$. Since $O_1 \subseteq O_2$, $\mathrm{st}_n(\gamma;O_1) \subseteq \mathrm{st}_n(\gamma;O_2)$. Assume the equality holds; otherwise the result is proved. Then, since γ is a proper face of α, the result holds inductively for some $\beta \in \gamma(O_1;O_2) \subseteq \alpha(O_1;O_2)$. $\square$

Remark 4.6. The previous lemma provides the key to prove that any O'-component C of O is itself an O'-connected object. For this, one observes that $\mathrm{st}_n(\alpha;O) \subseteq C$ for any cell $\alpha \in K$ with $f(O \cup O',\alpha) = 1$ and for which there exists $\sigma \in C$ with $\alpha \leqslant \sigma$. Hence $\mathrm{st}_n(\alpha;C \cup O') = \mathrm{st}_n(\alpha;O \cup O')$ and, moreover, $f(C \cup O',\alpha) = 1$ since otherwise Lemma 4.5 leads to a contradiction. From this, it is easily derived that any O'-path in O which is contained in C is also an O'-path in C, and the result follows.

For the next lemma from graph theory, we recall that a vertex v in a directed graph $G(V,E)$ is called a *source* if none edge in G ends at v (i.e., $\{w \mid (w,v) \in E\} = \emptyset$), and v is a *sink* if none edge in G starts from v (i.e.,$\{w \mid (v,w) \in E\} = \emptyset$).

Lemma 4.7. *Let $G(V,E)$ be an acyclic, transitive and locally finite directed graph. For any pair a,b of sinks in a connected component of G, there exists an undirected path $(v_i)_{i=0}^k$ in G from a to b, where $k = 2m$ is an even number, such that v_{2i} is a sink and v_{2i-1} is a source, for $i = 1,\dots,m$.*

Proof. Since a,b are vertices in a component of G, we can consider an undirected path $\Gamma = (v_i)_{i=0}^k$ in G from a to b such that v_i and v_j are adjacent if and only if $|i-j| = 1$. If, in addition, a and b are sinks of G, then the length of Γ is necessarily an even number $k = 2m$. Indeed, since G is transitive, there are not three vertices v_{i-1}, v_i, v_{i+1} in Γ with $(v_{i-1},v_i),(v_i,v_{i+1}) \in E$. Otherwise, v_{i-1} and v_{i+1} would be adjacent. Similarly if (v_i,v_{i-1}), (v_{i+1},v_i) are edges in G.

Finally, we derive from Γ a new path for which the even vertices are sinks and the odd ones are sources. Indeed, if v_{2i} is not a sink, the set $\{v \mid (v_{2i},v) \in E\}$ is not empty and, furthermore, it contains a sink w_{2i} since G is acyclic, transitive and locally finite; then, we can replace v_{2i} with w_{2i} in Γ by the transitivity of G. Similarly, if v_{2i-1} is not a source, we replace it with the source w_{2i-1} that necessarily exists in the set $\{v \mid (v,v_{2i-1}) \in E\}$. $\square$

In the next lemma $\Gamma_1 * \Gamma_2$ will denote the concatenation of the paths $\Gamma_1 = (\gamma_i^1)_{i=0}^{m_1}$ and $\Gamma_2 = (\gamma_i^2)_{i=0}^{m_2}$, where $\gamma_{m_1}^1 = \gamma_0^2$; namely $\Gamma_1 * \Gamma_2 = (\gamma_0^1,\dots,\gamma_{m_1}^1,\gamma_1^2,\dots,\gamma_{m_2}^2)$.

Lemma 4.8. *Let $O_1 \subseteq O_2$ be two objects in a digital space (K,f) and $\alpha \in K$ such that $f(O_1,\alpha) = 0$, $f(O_2,\alpha) = 1$ and $st_n(\alpha;O_1) = st_n(\alpha;O_2)$. If $\beta_1,\beta_2 \in \alpha(O_1;O_2) = \{\gamma < \alpha \mid f(O_1,\gamma) = 0, f(O_2,\gamma) = 1\}$ are such that $st_n(\beta_i;O_1) \neq st_n(\beta_i;O_2)$, $i = 1,2$, then there* *exists a path* $\Gamma=(c(\gamma_i))_{i=0}^{2m}$ *in* $\mathscr{C}_{O_2}\backslash\mathscr{C}_{O_1}$ *from* $c(\beta_1)$ *to* $c(\beta_2)$ *with* $\gamma_{2j-1}\in O_2-O_1$, *for* $j=1,\dots,m$.

Proof. It is immediate from the hypothesis that $\mathrm{st}_n(\beta_i;O_1)$ is a proper subset of $\mathrm{st}_n(\beta_i;O_2)$. So that, there exist two n-cells $\sigma_{\beta_1},\sigma_{\beta_2}\in O_2-O_1$ such that $\beta_i<\sigma_{\beta_i}$, $i=1,2$. On the other hand, by property (5) in the definition of w.l.f.'s, $\alpha(O_1;O_2)$ is connected in $\partial\alpha$. Thus, there exists a path $\Theta=(c(\theta_i))_{i=0}^{k}$ in $\mathscr{C}_{O_2}\backslash\mathscr{C}_{O_1}$ from $c(\beta_1)$ to $c(\beta_2)$ with $\theta_i\in\alpha(O_1;O_2)$, $i=0,\dots,k$. The desired path Γ will be derived from Θ by induction on $\dim\alpha$.

Firstly, notice that $\dim\alpha>0$ by Lemma 4.4. Furthermore, $\dim\alpha>1$ since otherwise β_1 and β_2 are the vertices of the edge α, and $\{\beta_1,\beta_2\}=\alpha(O_1;O_2)$ would not be connected in $\partial\alpha$. So, assume $\dim\alpha=2$, and hence $\dim\theta_i\leqslant 1$ for $0\leqslant i\leqslant k$. We claim that there exists an n-cell $\sigma_i\in O_2-O_1$ such that $\theta_i<\sigma_i$, for each edge θ_i, $0<i<k$. Indeed, $\theta_{i-1},\theta_{i+1}$ are the vertices of θ_i and, since $\{\theta_{i-1},\theta_{i+1}\}=\theta_i(O_1;O_2)$ is not connected in $\partial\theta_i$, property (5) in the definition of w.l.f.'s implies that $\mathrm{st}_n(\theta_i;O_1)$ is a proper subset of $\mathrm{st}_n(\theta_i;O_2)$. Then, the path $\Gamma=\Gamma_1*\Gamma_2*\Gamma_3$ is obtained as follows: one replaces each edge θ_i in $(c(\theta_i))_{i=1}^{k-1}$ by the corresponding n-cell σ_i to obtain Γ_2; if $\dim\beta_1=0$ then $\Gamma_1=(c(\beta_1),c(\sigma_1))$ and otherwise, if $\dim\beta_1=1$, $\Gamma_1=(c(\beta_1),c(\sigma_{\beta_1}),c(\theta_1))$; and, similarly, either $\Gamma_3=(c(\sigma_{k-1}),c(\beta_2))$, if $\dim\beta_2=0$, or $\Gamma_3=(c(\theta_{k-1}),c(\sigma_{\beta_2}),c(\beta_2))$ for $\dim\beta_2=1$. Notice that, in any case, the length of Γ is an even number.

Next, assume the result holds for any cell β with $\dim\beta<l$ and let $\dim\alpha=l$. Since $\mathscr{C}_{O_2}\backslash\mathscr{C}_{O_1}$ is a transitive directed graph we can assume, as in the proof of Lemma 4.7, that there are not three successive points $c(\theta_{i-1}),c(\theta_i),c(\theta_{i+1})$ in Θ such that either $\theta_{i-1}<\theta_i<\theta_{i+1}$ or $\theta_{i-1}>\theta_i>\theta_{i+1}$. Let us consider the path $\Theta'=\Theta_1*\Theta*\Theta_2$, where Θ_1 is either $(c(\beta_1))$ if $\beta_1=\theta_0<\theta_1$ or otherwise, if $\theta_0>\theta_1$, $\Theta_1=(c(\beta_1),c(\sigma_{\beta_1}),c(\theta_1))$, and similarly either $\Theta_2=(c(\beta_2))$ if $\beta_2=\theta_k<\theta_{k-1}$ or $\Theta_2=(c(\theta_{k-1}),c(\sigma_{\beta_2}),c(\beta_2))$ for $\theta_k>\theta_{k-1}$. Recall that $\sigma_{\beta_i}\in O_2-O_1$ and $\beta_i<\sigma_{\beta_i}$ for $i=1,2$. In any case, notice that the length of Θ' is an even number. Thus one can write $\Theta'=(c(\theta_i'))_{i=0}^{2m}$; and it is easy to observe that the relations $\theta_{2j-2}'<\theta_{2j-1}'>\theta_{2j}'$ hold for each $1\leqslant j\leqslant m$.

From Θ' another path $\Gamma'=(c(\gamma_i'))_{i=0}^{2m}$ is derived as follows: $\gamma_{2j-1}'=\theta_{2j-1}'$ for $1\leqslant j\leqslant m$; and γ_{2j}', $0\leqslant j\leqslant m$, is either θ_{2j}' if $\mathrm{st}_n(\theta_{2j}';O_1)\neq\mathrm{st}_n(\theta_{2j}';O_2)$ or, otherwise, $\gamma_{2j}'\in\theta_{2j}'(O_1;O_2)$ is the face of θ_{2j}' whose existence is granted by Lemma 4.5. Notice that $\mathrm{st}_n(\gamma_{2j}';O_1)\neq\mathrm{st}_n(\gamma_{2j}';O_2)$ in any case, and the relations $\gamma_{2j-2}'<\gamma_{2j-1}'>\gamma_{2j}'$ still hold. Two cases are then possible for each γ_{2j-1}':

Case 1: $\mathrm{st}_n(\gamma_{2j-1}';O_1)\neq\mathrm{st}_n(\gamma_{2j-1}';O_2)$. Then there exists $\sigma_{2j-1}\in O_2-O_1$ with $\gamma_{2j-1}'\leqslant\sigma_{2j-1}$, and hence $\Gamma_j=(c(\gamma_{2j-2}'),c(\sigma_{2j-1}),c(\gamma_{2j}'))$ is a path in $\mathscr{C}_{O_2}\backslash\mathscr{C}_{O_1}$.

Case 2: $\mathrm{st}_n(\gamma_{2j-1}';O_1)=\mathrm{st}_n(\gamma_{2j-1}';O_2)$. As $\dim\gamma_{2j-1}'<\dim\alpha$, there exists inductively a path Γ_j from $c(\gamma_{2j-2}')$ to $c(\gamma_{2j}')$ satisfying the required properties.

Then, the desired path is $\Gamma=\Gamma_1*\Gamma_2*\cdots*\Gamma_m$. $\square$

Proposition 4.9. *Let* O *and* O' *be two disjoint digital objects in* (K,f). *Two* n*-cells* $\sigma,\tau\in O$ *belong to an* O'*-component of* O *if and only if their centroids* $c(\sigma),c(\tau)$ *are vertices of a component of* $\mathscr{C}_{O\cup O'}\backslash\mathscr{C}_{O'}$.

Proof. First, assume that $\sigma, \tau \in O$ belong to the same O'-component of O, and let $(\sigma_i)_{i=0}^{m} \subseteq O$ be an O'-path from σ to τ. Then, the sequence $(c(\tau_i))_{i=0}^{2m}$, where $\tau_{2i} = \sigma_i$ $(0 \leqslant i \leqslant m)$ and $\tau_{2i-1} \leqslant \sigma_{i-1} \cap \sigma_i$ is such that $f(O \cup O', \tau_{2i-1}) = 1$ and $f(O', \tau_{2i-1}) = 0$ $(1 \leqslant i \leqslant m)$ is a path in $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$. Notice that in the special case $O' = \emptyset$, the complement $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$ coincides with the conceptual level $\mathscr{C}_O$ and $(c(\tau_i))_{i=0}^{2m}$ turns to be a path in $\mathscr{C}_O$.

Conversely, we claim that there is a path $\Gamma = (c(\gamma_i))_{i=0}^{2m}$ in $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$ from $c(\sigma)$ to $c(\tau)$ with $\gamma_{2i} \in O$ $(0 \leqslant i \leqslant m)$. Then, $(\gamma_{2i})_{i=0}^{m}$ is an O'-path from σ to τ. Indeed, we know that $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$ is an acyclic, transitive and locally finite directed graph. Then, Lemma 4.7 applies and we find a path $(c(\sigma_i))_{i=0}^{2m}$ from $c(\sigma)$ to $c(\tau)$ such that $c(\sigma_{2i})$ is a sink and $c(\sigma_{2i-1})$ is a source in $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$, for $0 \leqslant i \leqslant m$.

Next, we will derive Γ from this other path. For this consider first the case $O' = \emptyset$. Since $f(O, \sigma_{2i}) = 1$, property (2) of w.l.f.'s yields that $\sigma_{2i} \in \mathrm{supp}(O)$, and hence it is face of some $\tau_i \in O$. But σ_{2i} cannot be a proper face of τ_i since $c(\sigma_{2i})$ is a sink in $\mathscr{C}_O$; thus, $\sigma_{2i} = \tau_i \in O$ and hence $\Gamma = (c(\sigma_i))_{i=0}^{2m}$ is the desired path.

If $O' \neq \emptyset$, Lemma 4.5 implies that $\mathrm{st}_n(\sigma_{2i-1}; O') \neq \mathrm{st}_n(\sigma_{2i-1}; O \cup O')$ since $c(\sigma_{2i-1})$ is a source. Moreover, if $\sigma_{2i} \notin O$, for some $1 \leqslant i < m$, then $\mathrm{st}_n(\sigma_{2i}; O') = \mathrm{st}_n(\sigma_{2i}; O \cup O')$; otherwise, $c(\sigma_{2i})$ would not be a sink. Thus, by Lemma 4.8, there exists a path $\Gamma_i = (c(\gamma_j^i))_{j=0}^{2m_i}$ in $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$ from $c(\sigma_{2i-1})$ to $c(\sigma_{2i+1})$ with $\gamma_{2j-1}^i \in O$. Then $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_m$, where $\Gamma_i = (c(\sigma_{2i-1}), c(\sigma_{2i}), c(\sigma_{2i+1}))$ if $\sigma_{2i} \in O$. $\square$

We are now ready to characterize O'-connectedness at the conceptual level of our architecture:

Theorem 4.10. *There is a* 1–1 *map between the O'-components of O and the components of $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$. Moreover, the O'-components of O are the objects $O_G = \{\sigma \in O \mid c(\sigma)$ is a vertex of $G\}$, where G ranges over the components of $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$.*

Proof. The result follows from Proposition 4.9 if we show that $O_G \neq \emptyset$ for each component G of $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$. Let $c(\tau)$ be a vertex in G. If $\mathrm{st}_n(\tau; O')$ is a proper subset of $\mathrm{st}_n(\tau; O \cup O')$ we trivially find an n-cell $\sigma \in O$ with $\tau \leqslant \sigma$. Notice that this is the case when $O' = \emptyset$, since $\mathrm{st}_n(\tau; O) \neq \emptyset = \mathrm{st}_n(\tau; \emptyset)$ whenever $c(\tau)$ is a vertex of $\mathscr{C}_O$. If $\mathrm{st}_n(\tau; O') = \mathrm{st}_n(\tau; O \cup O')$, we find $\tau' \in \tau(O'; O \cup O')$ with $\mathrm{st}_n(\tau'; O') \neq \mathrm{st}_n(\tau'; O \cup O')$ by Lemma 4.5. Then, as in the previous case, $\tau' \leqslant \sigma \in O$, and the result follows since $c(\tau'), c(\tau)$ and $c(\sigma)$ are all vertices of G. $\square$

The characterization of O'-connectedness at the simplicial level of our architecture (Theorem 4.2(2)) is an immediate consequence of Theorem 4.10 since $\mathscr{C}_{O \cup O'} \backslash \mathscr{C}_{O'}$ is the 1-skeleton of $\mathscr{A}_{O \cup O'} \backslash \mathscr{A}_{O'}$. So that, when $O' = \emptyset$, the components of a digital object O are characterized by the components of its simplicial analogue $\mathscr{A}_O$. This leads us immediately to the characterization at the continuous level, since the latter are in 1–1 correspondence with the components of the continuous analogue $|\mathscr{A}_O|$. But in the general case, $O' \neq \emptyset$, the result is not so simple because $|\mathscr{A}_{O \cup O'}| - |\mathscr{A}_{O'}|$ is an open polyhedron. To complete the characterization at this level we need some additional results from polyhedral topology.

Proposition 4.11. *Let $L_1 \neq L_2$ be full subcomplexes of a simplicial complex L. Then two vertices $v, w \in L_1 \backslash L_2$ which are joined by a path in $|L_1| - |L_2|$ are joined by a simplicial path in $L_1 \backslash L_2$; that is, there is a sequence $(v_i)_{i=0}^m$ of vertices such that $v_0 = v$, $v_m = w$ and $\langle v_{i-1}, v_i \rangle$ is a* 1*-simplex in $L_1 \backslash L_2$, for $i = 1, \ldots, m$.*

In the proof of Proposition 4.11 we use the following lemma. Recall that the open star of a vertex $v \in L$ is the open set $\overset{\circ}{\mathrm{st}}(v; L) = \bigcup \{\overset{\circ}{\sigma} \mid v \in \sigma \in L\} \subseteq |L|$.

Lemma 4.12. *For $L_1, L_2 \subseteq L$ as above, the following equality holds*:

$$|L_1| - |L_2| = \bigcup \{\overset{\circ}{st}(v; L_1) \mid v \text{ is a vertex in } L_1 \backslash L_2\}.$$

Proof. As $L_1, L_2 \subseteq L$ are full subcomplexes, the set of vertices of $L_1 \backslash L_2$ is non-empty unless $L_1 \subseteq L_2$. Then, if v is a vertex in $L_1 \backslash L_2$, it is not difficult to show that $\overset{\circ}{\mathrm{st}}(v; L_1) \subseteq |L_1| - |L_2|$. For this it is enough to check that if $x \in \overset{\circ}{\mathrm{st}}(v; L_1) \cap |L_2|$ then there exists a simplex $\sigma \in L_2$ with $x \in \overset{\circ}{\sigma}$ and $v \in \sigma$, which is a contradiction. Conversely, assume that $x \in |L_1| - |L_2|$ and let σ be the only simplex in $L_1 \subseteq L$ such that $x \in \overset{\circ}{\sigma}$. Hence $\sigma \notin L_2$ and, as L_2 is a full subcomplex, there exists a vertex v of σ which does not belong to L_2. Thus, $x \in \overset{\circ}{\sigma} \subseteq \overset{\circ}{\mathrm{st}}(v; L_1)$ and the result follows. $\square$

Proof of Proposition 4.11. Let $\Gamma : I = [0,1] \to |L_1| - |L_2|$ be a continuous path with $\Gamma(0) = v$ and $\Gamma(1) = w$. We can apply the Lebesgue Lemma to the open cover $\{\Gamma^{-1}(\overset{\circ}{\mathrm{st}}(v; L_1)) \mid v \notin L_2\}$ of I. Thus, there is a partition of the unit interval $[0, 1/n], \ldots, [k/n, (k+1)/n], \ldots, [(n-1)/n, 1]$ such that $\Gamma([k/n, (k+1)/n]) \subseteq \overset{\circ}{\mathrm{st}}(v_k; L_1)$ for some vertex v_k in $L_1 \backslash L_2$ $(0 \leqslant k < n)$. As v_k is the only vertex in $\overset{\circ}{\mathrm{st}}(v_k; L_1)$ it follows that $v_0 = v$ and $v_{n-1} = w$. Moreover, since $\Gamma((k+1)/n) \in \overset{\circ}{\mathrm{st}}(v_k; L_1) \cap \overset{\circ}{\mathrm{st}}(v_{k+1}; L_1)$, the edge $\langle v_k, v_{k+1} \rangle$ exists in $L_1 \backslash L_2$ $(0 \leqslant k < n)$ and the proof is complete. $\square$

Corollary 4.13. *Let $L_1, L_2 \subseteq L$ be full subcomplexes. Then the path-components of $|L_1| - |L_2|$ are in* 1–1 *correspondence with the components of $L_1 \backslash L_2$.*

Proof. As $L_1, L_2 \subseteq L$ are full subcomplexes the set of vertices of $L_1 \backslash L_2$ is non-empty unless $L_1 \subseteq L_2$. Also it is clear that each component of $C \subseteq L_1 \backslash L_2$ determines a unique path-component $D_C \subseteq |L_1| - |L_2|$ with $C \subseteq D_C$. In fact, $D_C = D_{C'}$ yields $C = C'$. This follows from Proposition 4.11.

Moreover, given a path-component $D \subseteq |L_1| - |L_2|$, let $x \in D$ and $\sigma \in L_1$ with $x \in \overset{\circ}{\sigma}$. As L_2 is a full subcomplex, let $\tau \in L_2$ be the face of σ spanned by the vertices of σ in L_2. Furthermore, there exists at least a vertex $v \in \sigma - \tau$. Then $v \in L_1 \backslash L_2$ and the segment $\Gamma \subseteq \sigma$ joining v to x is contained in $|L_1| - |L_2|$. Thus, if $C = C_v$ is the component of v in $L_1 \backslash L_2$ we have $D = D_C$ and the correspondence $C \to D_C$ is bijective. $\square$

Theorem 4.14. *There is a* 1–1 *map between the O'-components of O and the components of $|\mathscr{A}_{O\cup O'}| - |\mathscr{A}_{O'}|$. Moreover, the O'-components of O are the digital objects $O_X = \{\sigma \in O \mid c(\sigma) \in X\}$ where X ranges over the set of components of $|\mathscr{A}_{O\cup O'}| - |\mathscr{A}_{O'}|$.*

Proof. Since $\mathscr{A}_{O'}$ and $\mathscr{A}_{O\cup O'}$ are locally finite complexes, connectedness and path-connectedness are equivalent in the space $|\mathscr{A}_{O\cup O'}| - |\mathscr{A}_{O'}|$. Then, the result follows from Theorem 4.2(2) and Corollary 4.13. □

Remark 4.15. (1) We have just shown in Theorem 4.2 that the O'-connectivity represents accurately the connectivity of the complements of digital objects. As it was remarked in Section 3, property (5) in Definition 3.2 is required for this purpose and, actually, it is essential for the proof of Theorem 4.2. Notice, however, that only the two first requirements stated in property (5) (i.e., that $\alpha(O_1; O_2)$ is a non-empty connected set in $\partial\alpha$) have been used in that proof (in Lemmas 4.5 and 4.8, respectively). The third requirement (i.e., $f(O, \beta) = 1$ for every $\beta \in \alpha(O_1; O_2)$ and $O_2 \subseteq O$) is only needed to ensure that the device level of any digital object $O \subseteq \text{cell}_n(K)$, $(K(O), f_O)$, is itself a digital space (see the definition of device level of an object after Example 3.4).

(2) It is worth to point out that, although the weak lighting functions can be understood as particular types of Kovalevsky's "face membership rules" [14], our definition of connectivity is slightly different from that considered by this author, which is the same normally used in abstract cell complexes. We refer to [3, Section 6] for a complete discussion on this point. Also in [3] it is stated the relationship between our notion of connectivity and the (α, β)-connectivity defined on $\mathbb{Z}^3$ within the graph-theoretical approach to digital topology.

5. Digital manifolds

In the previous section we have used our architecture to show that a merely combinatorial definition, as the notion of O'-connectivity, is a suitable counterpart of the topological notion we have at the continuous level. But, as it was quoted in Section 3, one may also proceed along the inverse way; namely, given a continuous notion or result, one can try to translate it through the architecture to obtain its digital counterpart. Following this pattern we define the notion of digital manifold in terms of the continuous analogue as follows: an object M in a digital space (K, f) is called a *digital n-manifold* if its continuous analogue $|\mathscr{A}_M|$ is a polyhedral n-manifold without boundary (see [1]). However, doing that, the problem of characterizing digitally this notion naturally arises; that is, given an arbitrary digital space (K, f), find necessary and sufficient conditions at the logical level to determine what digital objects are digital manifolds in (K, f). Under this general setting, this "*digital characterization problem of manifolds*" seems to us extremely intricate. However, for dimension one we have obtained the following characterization of digital curves (1-manifolds), which shows that our notion coincides with the usual definition of curve in the graph-theoretical approach to digital topology.

Theorem 5.1. *Let M be a connected digital object containing at least four n-cells in a digital space (K, f). The two following properties are equivalent*:
1. *M is a digital 1-manifold.*
2. *Each $\sigma \in M$ is adjacent in M to exactly two other n-cells $\sigma_1, \sigma_2 \in M$.*

Proof. Assume M is a digital 1-manifold; that is, $|\mathscr{A}_M|$ is a polyhedral 1-manifold, and so $\mathrm{lk}(c(\alpha); \mathscr{A}_M) = \{c(\alpha_1), c(\alpha_2)\}$ consists of two vertices for each $c(\alpha) \in \mathscr{A}_M$. In particular, if $\alpha \in \mathrm{cell}_n(K)$ then $\alpha \in M$ and necessarily $\alpha_i < \alpha$ $(i = 1, 2)$; moreover, α_i is not a face of α_j, $\{i, j\} = \{1, 2\}$, since otherwise the 1-simplex $\langle c(\alpha_i), c(\alpha_j)\rangle$ is contained in $\mathrm{lk}(c(\alpha); \mathscr{A}_M)$. In case $\alpha \notin \mathrm{cell}_n(K)$, $\mathrm{st}_n(\alpha; M) = \{\alpha_1, \alpha_2\}$ since $\alpha \in \mathrm{supp}(M)$ by property (2) in Definition 3.2. Thus, for a given $\sigma \in M$, $\mathrm{lk}(c(\sigma); \mathscr{A}_M) = \{c(\alpha_1), c(\alpha_2)\}$ with $\alpha_1, \alpha_2 < \sigma$, and $\mathrm{lk}(c(\alpha_i); \mathscr{A}_M) = \{c(\sigma), c(\sigma_i)\}$ where $\sigma_i \in M$ and $\alpha_i = \sigma \cap \sigma_i$ $(i = 1, 2)$. Hence σ is adjacent in M to both σ_1 and σ_2; and if $\tau \in M - \{\sigma\}$ is adjacent to σ then $\alpha_i \leqslant \sigma \cap \tau$ for some $i \in \{1, 2\}$, and so $\tau = \sigma_i$.

Conversely we first observe that there are not three n-cells $\sigma_1, \sigma_2, \sigma_3 \in M$ mutually adjacent. Otherwise, as M is connected, the existence of a fourth n-cell in M implies that one of these n-cells $\sigma_1, \sigma_2, \sigma_3$ is adjacent to at least three n-cells in M. Next, we check that $\mathrm{lk}(c(\alpha); \mathscr{A}_M)$ is a 0-sphere for all $c(\alpha) \in \mathscr{A}_M$. Indeed, if $\dim \alpha < n$ then $\mathrm{st}_n(\alpha; M) = \{\alpha_1, \alpha_2\}$ with $\alpha = \alpha_1 \cap \alpha_2$, since $\alpha \in \mathrm{supp}(M)$ and all the cells in $\mathrm{st}_n(\alpha; M)$ are mutually adjacent. If $\dim \alpha = n$ (i.e., $\alpha \in M$), let $\tau_1, \tau_2 \in M$ be the two n-cells adjacent to α through the faces $\alpha_i \leqslant \alpha \cap \tau_i$ $(i = 1, 2)$. Then, $\alpha_i \nleqslant \alpha_j$, $\{i, j\} = \{1, 2\}$, since otherwise these three n-cells would be mutually adjacent. Hence, $\mathrm{lk}(c(\alpha); \mathscr{A}_M) = \{c(\alpha_1), c(\alpha_2)\}$ and the proof is finished. $\square$

From a practical point of view, rather than tackling the general “digital characterization problem of manifolds” stated above, it seems to us more realistic to find answers to the following, and apparently simpler, twofold problem: (1) given a fixed digital space (K_0, f_0), find necessary and sufficient conditions at the logical level to determine whether a digital object is a digital n-manifold in (K_0, f_0); and (2) given a class $\mathscr{C}$ of digital objects in a device model K_0, define a w.l.f. f_0 in such a way that $\mathscr{C}$ is the set of digital n-manifolds in (K_0, f_0). In [3] we solve the second form of this problem for the classes of (α, β)-surfaces, $\alpha, \beta \in \{6, 18, 26\}$ and $(\alpha, \beta) \notin \{(6, 6), (18, 6), (18, 18)\}$, and, in this paper, the class of strong 26-surfaces is characterized as the class of digital 2-manifolds of the digital space (R^3, f^{BM}) given in Section 7.

Despite these difficulties, the general “digital characterization problem of manifolds” is not an obstacle to translate relevant continuous results, as the Jordan–Brouwer Separation Theorem, to the device level. In [1] we already stated a digital version of the Jordan–Brouwer Theorem at an earlier stage in our framework, in which lighting functions had not been defined yet and the only “continuous perception” implicitly used was that associated with the w.l.f. $f_{\max}$ in Example 3.3. This theorem plays a key role in the proof of our main results in Section 8. Due to this, and also to its relevance, we next prove it for the present version of our architecture. For this we need the following technical lemma.

Lemma 5.2. *Let (K,f) be a digital space. Then*
1. *For any finite digital object O in (K,f), its continuous analogue is a compact polyhedron.*
2. *Assume $|\mathscr{A}_K| = \mathbb{R}^m$ and let $X \subseteq \mathbb{R}^m$ be a compact set. Then, $O_X = \{\sigma \in \mathrm{cell}_n(K) \mid \sigma \cap X \neq \emptyset\}$ is a finite digital object in (K,f).*

Proof. (1) Let $K(O) = \{\alpha \in K \mid \alpha \leqslant \sigma \in O\}$ be the subcomplex of K induced by the cells in O. If O is finite, the polyhedron $|K(O)| = \bigcup\{\sigma \mid \sigma \in O\}$ is compact since it is the finite union of compact polytopes. Thus $|\mathscr{A}_O| \subseteq |K(O)|$ is compact since it is a closed polyhedron.

(2) For each n-cell $\sigma \in \mathrm{cell}_n(K)$, let us consider the open set

$$U_\sigma = \bigcup\{\overset{\circ}{\mathrm{st}}(c(\alpha); \mathscr{A}_K) \mid c(\alpha) \in \mathrm{st}(c(\sigma); \mathscr{A}_K)\}.$$

Notice that $c(\sigma)$ is the only centroid in U_σ such that $\dim \sigma = \dim K$. Then $\{U_\sigma\}_{\sigma \in O_X}$ is an open cover of X. Indeed, if $x \in X$ there exists a simplex $\gamma = \langle c(\gamma_0), \ldots, c(\gamma_k)\rangle \in \mathscr{A}_K$ such that $x \in \overset{\circ}{\gamma}$. Since $\gamma_k \leqslant \sigma$ for some $\sigma \in \mathrm{cell}_n(K)$, then $c(\gamma_k) \in \mathrm{st}(c(\sigma); \mathscr{A}_K)$ and $x \in \overset{\circ}{\gamma} \subseteq U_\sigma$. Finally, as X is compact there exists a finite cover $\{U_{\sigma_i}\}_{i=1}^r \subseteq \{U_\sigma\}_{\sigma \in O_X}$, and $O_X \subseteq \{\sigma \in \mathrm{cell}_n(K) \mid \sigma \cap \sigma_i, 1 \leqslant i \leqslant r\}$ is a finite object since K is a locally finite complex. $\square$

Theorem 5.3 (Digital Jordan–Brouwer Theorem). *Let (K,f) be a digital space such that $|\mathscr{A}_K| = \mathbb{R}^m$. If a digital object M in (K,f) is a connected digital $(m-1)$-manifold, then its complement $\mathrm{cell}_n(K) - M$ is divided into two M-components. Moreover, if M is finite then one of the M-components is also finite.*

Proof. If M is a connected digital $(m-1)$-manifold, its continuous analogue $|\mathscr{A}_M|$ is a polyhedral $(m-1)$-manifold without boundary and, furthermore, it is connected by Theorem 4.3. Thus, the continuous Jordan–Brouwer Theorem (see, for example, III.11.17 in [7]) yields that $|\mathscr{A}_K| - |\mathscr{A}_M| = \mathbb{R}^m - |\mathscr{A}_M|$ has two connected components, and the result follows by Theorem 4.14 since these components characterize the M-components of $\mathrm{cell}_n(K) - M$.

If, in addition, M is finite, then $|\mathscr{A}_M|$ is compact by Lemma 5.2(1). Hence, one of the components B of $\mathbb{R}^m - |\mathscr{A}_M|$ is bounded by $|\mathscr{A}_M|$ (see 8.3.6 in [16]), and the topological closure $\bar{B}$ of B coincides with $B \cup |\mathscr{A}_M|$ which is a compact set. Let $F = \{\sigma \in \mathrm{cell}_n(K) \mid c(\sigma) \in B\}$ be the M-component of $\mathrm{cell}_n(K) - M$ determined by B; see Theorem 4.2. Then, Lemma 5.2(2) yields that $F \subseteq \{\sigma \in \mathrm{cell}_n(K) \mid \sigma \cap \bar{B} \neq \emptyset\}$ is a finite object. $\square$

In the literature it can be found several digital versions of the Jordan–Brouwer Theorem for different families of digital surfaces defined on the grid $\mathbb{Z}^3$. We cite among others the papers of Morgenthaler and Rosenfeld [17], Kong and Roscoe [9], and Kopperman et al. [13]. Theorem 5.3 can be understood as a generalization of these results to arbitrary dimension and, furthermore, to grids of points distinct from $\mathbb{Z}^n$. The generalization of the result due to Kopperman et al. is immediate after noticing that Khalimsky's spaces [8] agree with the conceptual level of the digital spaces $(R^n, f_{\max})$. While the results of Morgenthaler and Rosenfeld and Kong and Roscoe follow from Theorem 5.3 since (α,β)-surfaces, for $(\alpha,\beta)\neq(6,6)$, are digital 2-manifolds for suitable digital spaces $(R^3, f_{\alpha\beta})$; see [3, Theorem 13].

6. Strong 26-surfaces

In [6, Definition 6] Bertrand and Malgouyres introduced a new class of surfaces for the grid $\mathbb{Z}^3$ endowed with the $(26,6)$-adjacency. These surfaces, called *strong* 26-*surfaces*, were originally defined in terms of the global notion of strong homotopy. Later in [15] these authors provide an equivalent local characterization of strong 26-surfaces within the family of strongly separating objects. In Section 9 we shall use this characterization in order to prove that the class of strong 26-surfaces is exactly the class of digital 2-manifolds for a suitable digital space (R^3, f^{BM}). To do it, in this section we appropriately restate in the language of our framework some notions and results from [15].

Given a digital object O in the standard cubical decomposition of the Euclidean 3-space, R^3, let $N_k(\sigma;O)$ denote the set of 3-cells $\tau\in O$ $(\sigma\neq\tau)$ which are k-adjacent to σ $(k=6,18,26)$. If $N_k(\sigma;O)\neq\emptyset$ we say that *σ is k-adjacent to O*. Notice that $N_{26}(\sigma;O)=\mathrm{st}_3^*(\sigma;O)-\{\sigma\}$. In addition, let $G_6(\sigma;O)$ denote the set of 3-cells $\tau\in N_{26}(\sigma;O)$ such that there exists a 6-path in $\mathrm{st}_3^*(\sigma;O)$ from σ to τ of length less than or equal to 2. Using the identification of the grid $\mathbb{Z}^3$ with the set $\mathrm{cell}_3(R^3)$ of 3-cells in R^3, the following equalities are immediate for any $\sigma\in O\subseteq\mathbb{Z}^3$:
1. $N_{26}(\sigma;\mathbb{Z}^3-O)=\mathrm{st}_3^*(\sigma;R^3)-O$,
2. $G_6(\sigma;\mathbb{Z}^3-O)=G_6(\sigma;\mathrm{st}_3^*(\sigma;R^3)-O)$.

Definition 6.1 (*Definitions* 6 *and* 7 *in* [15]). Let S be a 26-connected object in $\mathbb{Z}^3$. Then, S is said to be a *near strong* 26-*surface* if the following four properties hold for all $\sigma\in S$:
1. $N_{26}(\sigma;\mathbb{Z}^3-S)$ has exactly two 6-components A_1^σ and A_2^σ which are 6-adjacent to σ.
2. $G_6(\sigma;\mathbb{Z}^3-S)$ has exactly two 6-components.
3. For each $\tau\in N_{26}(\sigma;S)$ the sets $N_{26}(\tau;A_1^\sigma)$ and $N_{26}(\tau;A_2^\sigma)$ are non-empty.
4. For each $\tau\in N_6(\sigma;S)$ the sets $G_6(\sigma;A_1^\sigma\cup\{\tau\})$ and $G_6(\sigma;A_2^\sigma\cup\{\tau\})$ are 6-connected.

An object $O\subseteq\mathbb{Z}^3$ is said to be *strongly separating* if $\mathbb{Z}^3-O$ has two 6-components and, moreover, each $\sigma\in O$ is 6-adjacent to both components (see [6]). Then, strong 26-surfaces are characterized as follows.

Theorem 6.2 (Theorem 2 in [15]). *Let S be a 26-connected strongly separating object in $\mathbb{Z}^3$. Then S is a strong 26-surface if and only if it is a near strong 26-surface.*

Moreover, in [15] it is also proved.

Proposition 6.3 (Proposition 1 in [15]). *Any strong 26-surface is strongly separating.*

Although the notion of near strong 26-surface is local, the characterization given in Theorem 6.2 is not completely local because the notion of strongly separating is a global one. To obtain a truly local characterization it remains to show (by Proposition 6.3) that any near strong 26-surface is a strongly separating object. This will be a consequence of our Propositions 8.2 and 8.11 that, in this way, generalize the proof given in [15, Theorem 6] only for finite near strong 26-surfaces. To show their result, Malgouyres and Bertrand associate with each near strong 26-surface S a continuous analogue Σ_S which is, in fact, a triangulation of a surface embedded in $\mathbb{R}^3$. We shall use this surface to prove that any strong 26-surface S is a digital 2-manifold in the digital space (R^3, f^{BM}). So, in order to introduce Σ_S, we recall more notations and results from [15] in the following paragraphs.

An *elementary cube* is a closed unit cube with vertices in $\mathbb{Z}^3$. Notice that the vertices of an elementary cube C are the centroids of eight 3-cells in R^3 which share a common 0-cell, and this 0-cell is the center of C. An elementary cube C is said to be *maximal* with respect to an object O if $C \cap O \neq \emptyset$ and $C \cap O = C' \cap O$ whenever $C \cap O \subseteq C' \cap O$ for some other elementary cube C'. For a fixed digital object O, and if there is no place to confusion, we will simply say that C is maximal. Finally, a maximal cube C is said to be *simple* if for any 3-cell σ with $c(\sigma) \in C \cap O$ the difference $C \cap \mathbb{Z}^3 - O$ consists of (centroids of) 3-cells in 6-components of $\mathrm{st}_3^*(\sigma; R^3) - O$ which are 6-adjacent to σ; otherwise C is said *non-simple*.

Lemma 6.4 (Lemmas 3 and 4 in [15]). *Let S be a near strong 26-surface. If $B \subseteq st_3^*(\sigma_0; R^3) - S$ is a 6-component which is not 6-adjacent to σ_0 (i.e., $A_i^{\sigma_0} \neq B$, $i = 1, 2$), then*:

1. $B = \{\tau_0\}$ *with* $\dim \sigma_0 \cap \tau_0 = 0$; *and*,
2. *if C is the elementary cube containing σ_0 and τ_0, then $C \cap \mathbb{Z}^3 - S = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ with $\dim \sigma_0 \cap \tau_i = 2$ for $i \neq 0$; that is, the three 6-adjacent 3-cells to σ_0 in C lie in $\mathbb{Z}^3 - S$ and the three 18-adjacent 3-cells to σ_0 in C lie in S. Moreover $C \cap A_i^{\sigma_0} \neq \emptyset$ $(i = 1, 2)$.*

Remark 6.5. From Lemma 6.4 it follows that C is maximal with respect to S, and this is the only possible non-simple maximal cube up to rotation or symmetry (see Fig. 5).

The proof of Lemma 6 in [15] actually shows the following.

Lemma 6.6 (Lemma 6 in [16]). *With the same notation as in Lemma 6.4, assume that $A_1^{\sigma_0}$ is the component of $st_3^*(\sigma_0; R^3) - S$ containing two 3-cells of $C \cap \mathbb{Z}^3 - S$; say τ_1 and τ_2. Then, $st_3^*(\alpha; S) \subseteq C$ for $\alpha = \tau_1 \cap \tau_2$ (see Fig. 5).*

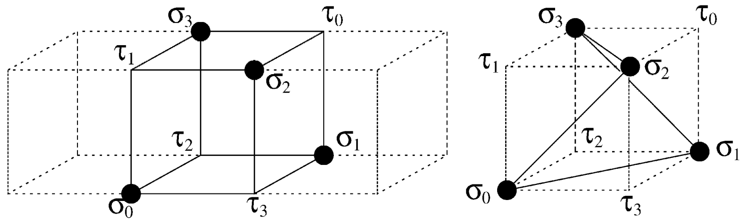


Fig. 5. A non-simple maximal elementary cube and its canonical cycle.

Lemma 6.7 (Lemma 7 in [15]). *With the same notation as in Lemmas 6.4 and 6.6, there exists a* 6*-path in* $st_3^*(\beta;R^3)-S$ *from* τ_0 *to* τ_3, *where* $\beta=\tau_0\cap\tau_3$. *Moreover,* $st_3^*(\beta;S)\subseteq C$ *by Lemma 6.6, since* τ_0 *and* τ_3 *are in a* 6*-component of* $st_3^*(\sigma_1;R^3)-S$ (*see Fig. 5*).

By the use of the previous lemmas, Malgouyres and Bertrand associate a *canonical cycle* with each non-simple maximal cube C, as the cycle defined by $(\sigma_0,\sigma_1,\sigma_3,\sigma_2,\sigma_0)$ in Fig. 5. Moreover, they show in [15, Lemma 8] that this cycle does not depend on the cell σ_i.

Remark 6.8. If C is a non-simple maximal elementary cube, the definition above yields that $\sigma_i,\sigma_j\in C\cap S$ are not successive in the canonical cycle of C if and only if $\mathrm{st}_3^*(\sigma_i\cap\sigma_j;S)\subseteq C$.

Each simple maximal cube C has also associated a canonical cycle. This is defined as the subgraph induced by the 3-cells in $C\cap S$ in the $1d$-adjacency graph of S. Recall that two 3-cells $\sigma,\tau\in S$ are said to be $1d$*-adjacent* if they are 6-adjacent or they are 18-adjacent and no 3-cell in S is 6-adjacent to both σ and τ. The existence of such a canonical cycle is proved in [15, Lemma 5].

Given a near strong 26-surface S, Σ_S is then defined as a union of triangles. Each triangle $T\in\Sigma_S$ has as vertices two successive points of the canonical cycle of C, where C is a maximal elementary cube with respect to S containing T; and its third vertex is either the center of the common 2-face of C and C', if there exists a maximal elementary cube $C'\neq C$ such that $C'\cap S=C\cap S$, or the center of C otherwise. In [15, Theorem 5] it is proved that the simplicial complex Σ_S is a topological surface without boundary embedded in $\mathbb{R}^3$.

7. The digital space (R^3, f^{BM})

In this section we introduce the digital space (R^3, f^{BM}) and analyze several general connectivity properties of objects in this space. In Section 9 we will show that the digital 2-manifolds in this space are exactly the strong 26-surfaces.

The device model of (R^3, f^{BM}) is the standard cubical decomposition of the Euclidean 3-space, R^3; and the function $f^{BM}:\mathscr{P}(\mathrm{cell}_3(R^3))\times R^3\to\{0,1\}$ is given by

$f^{BM}(O,\alpha)=1$ if and only if: (a) $\dim\alpha=3$ and $\alpha\in O$, (b) $\dim\alpha=0,2$ and $\alpha\in\operatorname{supp}(O)$, (c) $\dim\alpha=1$ and one of the following conditions holds:
(c1) $\mathrm{st}_3(\alpha;R^3)\subseteq O$,
(c2) $\alpha\in\operatorname{supp}(O)$ and $\mathrm{st}_3^*(\alpha;O)=\mathrm{st}_3(\alpha;O)$,
(c3) $\alpha\in\operatorname{supp}(O)$ and there exist $\sigma,\tau\in\mathrm{st}_3^*(\alpha;O)$, with $\sigma\cap\tau=\emptyset$.

In the next remark we collect some immediate properties of the function f^{BM}.

Remark 7.1. (1) Given a cell $\alpha\in R^3$, $f^{BM}(O,\alpha)=1$ for any digital object O such that $\mathrm{st}_3(\alpha;R^3)\subseteq O$. In particular, $f^{BM}(\mathrm{cell}_3(R^3),\alpha)=1$ for any cell $\alpha\in R^3$.

(2) Let O be a digital object in R^3 and let $\alpha<\sigma\in O$ with $f^{BM}(O,\alpha)=0$. If $\dim\alpha=2$ then $\alpha=\sigma\cap\tau$ with τ a 3-cell in $\mathrm{cell}_3(R^3)-O$. Moreover if $\dim\alpha=0$ then the definition of f^{BM} yields that $\alpha\notin\operatorname{supp}(O)$, and necessarily the elementary cube C with center in $c(\alpha)=\alpha$ contains a 2-face whose four vertices are (the centroids of) 3-cells in $\mathrm{cell}_3(R^3)-O$. Finally, if $\dim\alpha=1$ two cases are possible:

Case a: $\alpha\notin\operatorname{supp}(O)$. Then $c(\alpha)$ is the center of a 2-face F common to two elementary cubes, and the vertices of an edge of F are (centroids of) two 3-cells in $\mathrm{cell}_3(R^3)-O$.

Case b: $\alpha\in\operatorname{supp}(O)$. Let $v_1,v_2\in R^3$ be the vertices of α. As $f^{BM}(O,\alpha)=0$, the definition of f^{BM} implies that $\mathrm{st}_3(\alpha;O)$ is strictly contained in $\mathrm{st}_3^*(\alpha;O)=\mathrm{st}_3(v_i;O)$, for some $i\in\{1,2\}$. Then, the equalities $\mathrm{st}_3^*(\alpha;O)=\bigcup_{i=1}^2\mathrm{st}_3(v_i;O)$ and $\mathrm{st}_3(\alpha;O)=\bigcap_{i=1}^2\mathrm{st}_3(v_i;O)$ (see Lemma 3.1) yield $f^{BM}(O,v_1)+f^{BM}(O,v_2)=1$.

Next result shows that the pair (R^3,f^{BM}) is actually a digital space.

Proposition 7.2. *The function f^{BM} is a weak lighting function on R^3.*

Proof. Properties (1) and (2) in Definition 3.2 are obvious for f^{BM}, while Remark 7.1(1) implies trivially property (3). Also a straightforward checking shows property (4). For this one uses the equality $\mathrm{st}_n^*(\alpha;\mathrm{st}_n^*(\alpha;O))=\mathrm{st}_n^*(\alpha;O)$ in Lemma 3.1.

In order to check property (5) we consider objects $O_1\subseteq O_2$ and a cell $\alpha\in R^3$ with $\mathrm{st}_3(\alpha;O_1)=\mathrm{st}_3(\alpha;O_2)$, $f^{BM}(O_1,\alpha)=0$ and $f^{BM}(O_2,\alpha)=1$. Notice that $\alpha\in\operatorname{supp}(O_2)$ if and only if $\alpha\in\operatorname{supp}(O_1)$. Moreover, $\alpha\in\operatorname{supp}(O_2)$ since $f^{BM}(O_2,\alpha)=1$. Then, the definition of f^{BM} yields $\dim\alpha=1$ since otherwise $f^{BM}(O_1,\alpha)=1$. Moreover, $f^{BM}(O_1,\alpha)=0$ implies $\mathrm{st}_3(\alpha;R^3)\nsubseteq O_1$ and hence $\mathrm{st}_3(\alpha;R^3)\nsubseteq O_2$. In fact, we are under case (c3) in the definition of f^{BM} for the pair (O_2,α). Otherwise the equality $\mathrm{st}_3^*(\alpha;O_2)=\mathrm{st}_3(\alpha;O_2)$ leads to $f^{BM}(O_1;\alpha)=1$. Let $\sigma_1,\sigma_2\in\mathrm{st}_3^*(\alpha;O_2)$ with $\sigma_1\cap\sigma_2=\emptyset$ given by condition (c3), and let $\beta_i=\sigma_i\cap\alpha$, $i=1,2$, the vertices of α. As $\alpha\in\operatorname{supp}(O_2)$ it is readily checked that $\beta_i\in\operatorname{supp}(O_2)$, and hence $f^{BM}(O_2,\beta_i)=1$, $i=1,2$. Therefore $\alpha(O_1;O_2)$ is either $\{\beta_1\}$ or $\{\beta_2\}$ by case (b) in Remark 7.1(2).

Finally, for any object O with $O_2\subseteq O$ we have $f^{BM}(O,\beta_i)=1$, for $\{\beta_i\}=\alpha(O_1;O_2)$, since $f^{BM}(O_2,\beta_i)=1$ and so $\beta_i\in\operatorname{supp}(O_2)\subseteq\operatorname{supp}(O)$. $\square$

Our next goal is to show that the w.l.f. f^{BM} provides the $(26,6)$-connectivity on R^3.

Lemma 7.3. *Let O be an object in (R^3, f^{BM}). Two 3-cells $\sigma, \tau \in O$ are adjacent in O (i.e., their centroids define an edge in the logical level $\mathscr{L}_O$) if and only if $\sigma \cap \tau \neq \emptyset$; that is, σ and τ are 26-adjacent.*

Proof. One observes that $\alpha = \sigma \cap \tau \in \text{supp}(O)$ and hence $f^{BM}(O, \alpha) = 1$ when $\dim \alpha = 0, 2$. Moreover, if $\dim \alpha = 1$ and $f^{BM}(O, \alpha) = 0$ then, by Remark 7.1(2), $f^{BM}(O, \beta) = 1$ for a vertex β of α, and so σ and τ are adjacent in O. □

Proposition 7.4. *The w.l.f. f^{BM} provides the $(26, 6)$-connectivity on R^3. In other words, if O is a digital object in (R^3, f^{BM}), then*
1. *C is a connected component of O if and only if it is a 26-component; and,*
2. *C is a O-component of $\text{cell}_3(R^3) - O$ if and only if it is a 6-component.*

Proof. By Lemma 7.3 the paths in O are exactly the 26-paths and hence (1) holds.

In order to show (2) let C_σ and C'_σ be the O-component and the 6-component of $\sigma \in \text{cell}_3(R^3) - O$, respectively. We claim $C_\sigma = C'_\sigma$. For this let $\tau \in C'_\sigma$ and $(\sigma_i)_{i=0}^m \subseteq \text{cell}_3(R^3) - O$ a 6-path from σ to τ. For each cell $\alpha_i = \sigma_{i-1} \cap \sigma_i$ we have $f^{BM}(O, \alpha_i) = 0$ and hence $(\sigma_i)_{i=0}^m$ is an O-path in $\text{cell}_3(R^3) - O$ by Remark 7.1(1); see Definition 4.1.

Conversely, if $(\sigma_i)_{i=0}^m$ is an O-path from σ to τ and $\alpha_i \leqslant \sigma_{i-1} \cap \sigma_i$, $i = 1, \ldots, m$ is the face which makes σ_i and σ_{i-1} O-adjacent (i.e., $f^{BM}(O, \alpha_i) = 0$) we can assume that $\dim \alpha_i \leqslant 1$ since otherwise σ_i and σ_{i-1} are directly 6-adjacent. Assume $\dim \alpha_i = 0$. Then a 2-face of the elementary cube with center in $c(\alpha_i) = \alpha_i$ has its four vertices in $\text{cell}_3(R^3) - O$; see Remark 7.1(2). From this observation one readily finds a 6-path in $\text{cell}_3(R^3) - O$ from σ_{i-1} to σ_i. In case $\dim \alpha_i = 1$ and σ_{i-1} is not 6-adjacent to σ_i (that is, their centroids lie in a 2-face of an elementary cube) there are two possible cases. If $\text{st}_3(\alpha_i; O)$ has at most one element, it is clear that σ_{i-1} and σ_i are joined by a 6-path in $\text{cell}_3(R^3) - O$. Otherwise, if $\text{st}_3(\alpha_i; O)$ has exactly two elements, we use case (b) in Remark 7.1(2) to get $f^{BM}(O, \beta) = 0$ for a vertex $\beta < \alpha_i \leqslant \sigma_{i-1} \cap \sigma_i$, and then we are again in the case $\dim \alpha_i = 0$ above. □

Since the function f^{BM} satisfies $f^{BM}(\text{cell}_3(R^3), \alpha) = 1$ for all $\alpha \in R^3$, the continuous analogue of the whole space $|\mathscr{A}_{R^3}| = \mathbb{R}^3$ is the Euclidean 3-space. Hence we derive from Theorem 5.3 the following Jordan–Brouwer Theorem for digital 2-manifolds in (R^3, f^{BM}). In order to ease the writing, the digital 2-manifolds in (R^3, f^{BM}) will be called *f^{BM}-surfaces*.

Theorem 7.5. *Let S be a connected f^{BM}-surface. Then $\text{cell}_3(R^3) - S$ has two S-components (6-components according to Proposition 7.4). Moreover, if S is finite then one of the S-components is also finite.*

8. Characterizing f^{BM}-surfaces as near strong 26-surfaces

As it was pointed out in Section 3, the continuous analogue given in our architecture may be used to define new digital notions. From a practical point of view it arises the problem of characterizing those notions at a level so close to the device one as possible. This is the case of our definition of digital manifold (see Section 5). Although to obtain a general characterization of this notion seems to be a difficult problem, Bertrand–Malgouyres' near strong 26-surfaces provide such a characterization for digital 2-manifolds in the digital space (R^3, f^{BM}) as stated below. More explicitly

Theorem 8.1. *Let S be a connected digital object in (R^3, f^{BM}). Then S is an f^{BM}-surface if and only if it is a near strong 26-surface.*

This section is aimed to the proof of this theorem. The "only if" part is proved in Proposition 8.14 below. In order to prove the "if" part we make use of the continuous analogue Σ_S constructed by Malgouyres and Bertrand for each near strong 26-surface S. Since Σ_S is actually a surface without boundary embedded in $\mathbb{R}^3$ [15, Theorem 5] it will suffice to show that our simplicial analogue $\mathscr{A}_S$ triangulates Σ_S. Namely

Proposition 8.2. *Let S be a near strong 26-surface. Then the simplicial analogue $\mathscr{A}_S$ is a simplicial subdivision of Malgouyres–Bertrand's analogue Σ_S, and hence S is an f^{BM}-surface.*

In the following lemmas we obtain several technical results which are needed in the complex proof of Proposition 8.2 (done after Lemma 8.10). We start characterizing the cells $\alpha \in R^3$ with $c(\alpha) \in \mathscr{A}_S$ in terms of the maximal cubes containing $c(\alpha)$. This is done in Lemma 8.3 for $\dim \alpha = 0$ and in Lemma 8.4 for $\dim \alpha = 1$, since cases $\dim \alpha \in \{2, 3\}$ are obvious from the definition of f^{BM}.

Lemma 8.3. *Let O be a digital object in (R^3, f^{BM}). For an elementary cube C with $C \cap O \neq \emptyset$ the following properties are equivalent.*
1. *If C' is an elementary cube with $C \cap O \subseteq C' \cap O$ then $C = C'$, and hence C is maximal with respect to O.*
2. *If $\alpha \in R^3$ is the 0-cell such that $c(\alpha)$ is the center of C then $\alpha \in \operatorname{supp}(C \cap O)$, and hence $c(\alpha) \in \mathscr{A}_O$.*

Proof. Assume (2) and $C \cap O \subseteq C' \cap O$; then $\alpha \in \operatorname{supp}(C \cap O) \subseteq \operatorname{supp}(C' \cap O)$ and hence $c(\alpha)$ is also the center of C'. Thus, $C = C'$.

Conversely, if $\alpha \notin \operatorname{supp}(C \cap O)$ then $C \cap O$ is part of a face $F \subseteq C$ (see Remark 7.1 (2)). Then $C \cap O \subseteq C' \cap O$ for $C' \neq C$ the elementary cube with $C \cap C' = F$. $\square$

Lemma 8.4. *Let S be a near strong 26-surface and α an 1-cell in R^3. Then $c(\alpha) \in \mathscr{A}_S$ if and only if the two elementary cubes C_1, C_2 with $c(\alpha) \in C_1 \cap C_2$ are both maximal with respect to S and, moreover, one of the two following conditions holds:*
1. *$C_1 \cap S = C_2 \cap S = st_3(\alpha; R^3)$;*
2. *$\operatorname{st}_3(\alpha; S) = \{\sigma, \tau\}$ and σ, τ are successive in the canonical cycle of C_i $(i = 1, 2)$.*

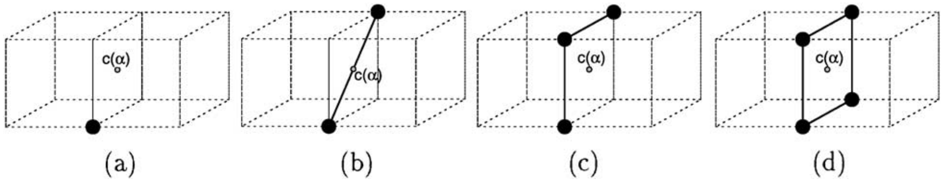


Fig. 6. Configurations of maximal cubes C_1, C_2, with respect to an object O, such that $C_1 \cap O = C_2 \cap O$. Only configuration (d) can appear in a near strong 26-surface.

The proof of Lemma 8.4 is immediate after the following four lemmas. To introduce them notice that, by the definition of f^{BM}, $c(\alpha) \in \mathscr{A}_S$ if only one of the following cases occurs:

(a) $\mathrm{st}_3(\alpha; R^3) \subseteq S$;
(b) $\alpha \in \mathrm{supp}(S)$ and $\mathrm{st}_3^*(\alpha; S) = \mathrm{st}_3(\alpha; S) \neq \mathrm{st}_3(\alpha; R^3)$;
(c) $\alpha \in \mathrm{supp}(S)$, $\mathrm{st}_3(\alpha; R^3) \nsubseteq S$ and there exist $\sigma_1, \sigma_2 \in \mathrm{st}_3^*(\alpha; S)$ with $\sigma_1 \cap \sigma_2 = \emptyset$.

Then, in Lemma 8.5 we shall prove that case (b) is not possible for a near strong 26-surface; Lemma 8.6 is used to prove Lemma 8.7, which shows that case (a) corresponds to (1) in Lemma 8.4; and, finally, case (c) corresponds to (2) as it is shown in Lemma 8.8.

Lemma 8.5. *With the notation of Lemma* 8.4, *assume that* $C_1 \cap S = C_2 \cap S$ *and both cubes* C_1, C_2 *are maximal. Then* $C_i \cap S = st_3(\alpha; R^3)$ $(i = 1, 2)$. *Therefore, for any* 1-*cell* $\alpha \in supp(S)$ *the case* $st_3^*(\alpha; S) = st_3(\alpha; S) \neq st_3(\alpha; R^3)$ *never occurs.*

Proof. In Fig. 6 are shown the four possible configurations of two maximal cubes C_1, C_2, with respect to an arbitrary object O, which share a common 2-face of center $c(\alpha)$ and such that $C_1 \cap O = C_2 \cap O$. If $O = S$ is a near strong 26-surface, then both C_1 and C_2 are simple cubes according to Lemma 6.4, and thus the subgraph induced by $C_i \cap S$ in the $1d$-adjacency graph of S must be a cycle by [15, Lemma 5]. Hence, the only possible configuration of C_1 and C_2 for S is that depicted in Fig. 6(d), and so $C_i \cap S = \mathrm{st}_3(\alpha; R^3)$.

The second part is now immediate since $\mathrm{st}_3^*(\alpha; S) = \mathrm{st}_3(\alpha; S)$ implies $C_1 \cap S = C_2 \cap S = \mathrm{st}_3(\alpha; S)$ and moreover, $\alpha \in \mathrm{supp}(S)$ yields that both C_1 and C_2 are maximal. $\square$

Lemma 8.6. *Let* C *be a maximal cube with respect to a near strong* 26-*surface* S. *Assume, in addition, that the* 0-*cell* $\beta \in R^3$ *is the center of* C, *and* $\alpha_i \in R^3$ $(1 \leqslant i \leqslant 6)$ *are the six* 1-*cells such that* $c(\alpha_i)$ *are the centers of the* 2-*faces of* C. *Then* $\beta \in \mathrm{supp}(S)$ *if and only if* $st_3(\alpha_i; R^3) \nsubseteq S$, *for* $1 \leqslant i \leqslant 6$.

Proof. Assume firstly that $\beta \in \mathrm{supp}(S)$. If, in addition, $\mathrm{st}_3(\alpha_i; R^3) \subseteq S$ for some $1 \leqslant i \leqslant 6$, then C is a simple maximal cube according Lemma 6.4. However, the subgraph induced by $C \cap S$ in the $1d$-adjacency graph of S is not a cycle, which leads us to a contradiction with [15, Lemma 5].

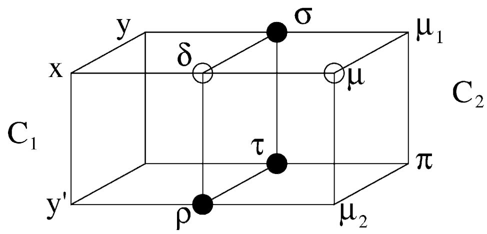


Fig. 7. The center of a 2-face of an elementary cube which has exactly three of its vertices in a near strong 26-surface S is not lighted for S.

Conversely, if $\beta \notin \mathrm{supp}(S)$ then $C\cap S$ is part of a 2-face $F_i \subseteq C$. Assume that $c(\alpha_i)$ is the center of F_i. Then, if $C' \neq C$ is the other elementary cube with $C\cap C' = F_i$, the maximality of C yields that $C\cap S = C'\cap S$ and, so, C' is also maximal. Then, Lemma 8.5 implies that $\mathrm{st}_3(\alpha_i;R^3) = C\cap S$, and the result follows. □

Lemma 8.7. *With the notation of Lemma* 8.4, $st_3(\alpha;R^3)\subseteq S$ *if and only if* $C_i\cap S = st_3(\alpha;R^3)$ $(i=1,2)$, *and so both* C_1 *and* C_2 *are maximal and simple.*

Proof. The inclusion $\mathrm{st}_3(\alpha;R^3)\subseteq C_i\cap S$ follows immediately from the hypothesis. Also the other inclusion holds. Otherwise the 0-cell β_i in the center of C_i lies in supp(S), since C_i contains elements of S on each face (see Remark 7.1(2)). But according Lemma 8.6 this is not possible. Finally, from the equalities $C_i\cap S = \mathrm{st}_3(\alpha;R^3)$ $(i=1,2)$ it is easily derived that both C_1 and C_2 are maximal cubes; moreover, they are simple by Lemma 6.4.

The converse is obvious. □

Lemma 8.8. *With the notation of Lemma* 8.4, *assume that* $st_3(\alpha;R^3)\nsubseteq S$. *Then* α *lies in* supp(S) *and there exist* $\sigma_1,\sigma_2\in st_3^*(\alpha;S)$ *with* $\sigma_1\cap\sigma_2=\emptyset$ *if and only if* C_1 *and* C_2 *are maximal and, moreover,* $st_3(\alpha;S)=\{\sigma,\tau\}$ *with* σ *and* τ *successive vertices in the canonical cycle of* C_i.

Proof. Assume $\alpha\in\mathrm{supp}(S)$, and so $\mathrm{st}_3(\alpha;S)$ contains at least two 3-cells. In addition, the existence of σ_1 and σ_2 yields that C_i is maximal by Lemma 8.3. Moreover, if $\mathrm{st}_3(\alpha;S)$ contains only two 3-cells they are $1d$-adjacent and successive in the canonical cycle of C_i (being C_i simple or not). For this we use Remark 6.8 and Lemma 5 in [15].

We finish the proof by showing that $\mathrm{st}_3(\alpha;S)$ does not contain three 3-cells (four 3-cells is not possible since $\mathrm{st}_3(\alpha;R^3)\nsubseteq S$). Assume for a moment $\mathrm{st}_3(\alpha;S)=\{\sigma,\tau,\rho\}$ and $\delta\in\mathrm{st}_3(\alpha;R^3)-S$ as in Fig. 7.

Property (3) in Definition 6.1 implies that ρ is 26-adjacent to the 6-components $A_i^\sigma\subseteq\mathrm{st}_3^*(\sigma;R^3)-S$ which are 6-adjacent to σ. Then there exists a vertex (3-cell) $w\in\mathrm{st}_3^*(\alpha;R^3)-S=(C_1\cup C_2)-S$ which does not lie in the 6-component of δ in $\mathrm{st}_3^*(\alpha;R^3)-S\subseteq\mathrm{st}_3^*(\sigma;R^3)-S$.

Without loss of generality we can assume that $w \in C_1$. Then $w \neq x$ (x is 6-adjacent to δ) and the other possibilities for $w \in C_1$ leads to $x \in S$ or $\{y, y'\} \subseteq S$. Hence there is no 6-path in $C_1 \cap \mathbb{Z}^3 - S$ from δ to τ. Nevertheless, δ is 18-adjacent to τ and so it belongs to either A_1^τ or A_2^τ by Lemma 6.4. Therefore one finds a 6-path in $(\mathrm{st}_3^*(\tau; R^3) - S) \cup \{\tau\}$ from δ to τ which necessarily goes through μ. Hence $\mu \notin S$. Next we show that $\mu_1, \mu_2 \notin S$ and so $\pi = \sigma_i \in S$ for $i = 1$ or 2 and there is no canonical cycle in the maximal cube C_2. Therefore $\mathrm{st}_3(\alpha; S)$ only contains two 3-cells.

To show $\mu_1, \mu_2 \notin S$ we assume on the contrary that $\mu_1 \in S$. Then we apply property (3) in Definition 6.1 to μ_1 and ρ and we get $\mu_2 \in S$. Hence $\delta \notin A_1^\tau \cup A_2^\tau$ which is a contradiction.

Conversely, the maximality of C_i $(i = 1, 2)$ yields that $C_1 \cap S \neq C_2 \cap S$ by Lemma 8.5, and so the sets $(C_1 - C_2) \cap S$ and $(C_2 - C_1) \cap S$ contains 3-cells σ_1 and σ_2, respectively. Then $\sigma_1 \cap \sigma_2 = \emptyset$ and $\sigma_i \in \mathrm{st}_3^*(\alpha; S) = (C_1 \cup C_2) \cap S$. □

To proceed with the proof of Proposition 8.2, we next show that, for a given near strong 26-surface S, it is also a vertex of $\mathscr{A}_S$ the center of each edge in Σ_S which is determined by two successive vertices in the canonical cycle of a maximal cube with respect to S.

Lemma 8.9. *Let C be a maximal cube with respect to a near strong 26-surface S. If $\sigma, \tau \in S$ are successive vertices in the canonical cycle of C then $c(\sigma \cap \tau) \in \mathscr{A}_S$.*

Proof. Since σ, τ are successive in the canonical cycle of C we have $\dim \sigma \cap \tau \geqslant 1$. If $\dim \sigma \cap \tau = 2$ there is nothing to prove. Otherwise, if $\dim \sigma \cap \tau = 1$ we claim that $\mathrm{st}_3(\sigma \cap \tau; S) = \{\sigma, \tau\}$. Indeed, if C is simple this is derived from the definition of the canonical cycle of C since σ and τ are 18-adjacent, and if C is not simple it is immediate from Lemma 6.4.

Let C' be the other elementary cube such that $C \cap C'$ is the 2-face whose center is $c(\sigma \cap \tau)$. To finish the proof, it only remains, by Lemma 8.8, to show that C' is also a maximal cube. Otherwise, there is no canonical cycle in C', and thus the edge $\langle c(\sigma), c(\tau) \rangle$ lies in the boundary of exactly one triangle of Σ_S. But this is a contradiction since Σ_S has no boundary. □

Finally, in the proof of Proposition 8.2, we will also need the following.

Lemma 8.10. *Let S be a near strong 26-surface and $\alpha \in R^3$ a 2-cell with $st_3(\alpha; R^3) = \{\sigma, \tau\}$. If $c(\alpha) \in \mathscr{A}_S$ then $\sigma, \tau \in S$ are successive vertices in the canonical cycle of some maximal cube with respect to S.*

Proof. The definition of f^{BM} yields that $\sigma, \tau \in S$. In that case, there necessarily exists a maximal cube C containing $c(\sigma)$ and $c(\tau)$. Moreover, C is simple according Lemma 6.4. Thus, its canonical cycle is the subgraph induced by $C \cap S$ in the $1d$-adjacency graph of S; and hence σ, τ are successive in such a cycle since they are 6-adjacent. □

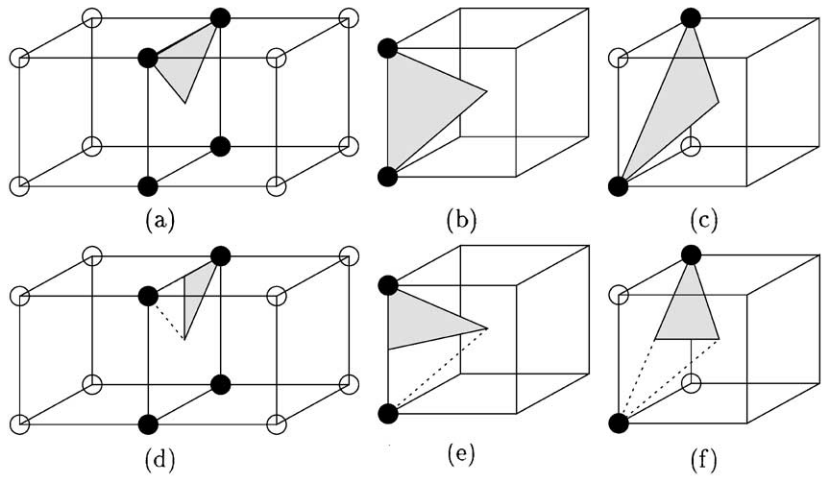


Fig. 8. The three possible configurations (up to rotation or symmetry) of the triangles in Σ_S (a)–(c), and their subdivision by triangles in $\mathscr{A}_S$ (d)–(f).

Proof of Proposition 8.2. It will suffice to show (1) any triangle $T \in \Sigma_S$ is the union of two triangles of $\mathscr{A}_S$ and (2) any simplex of $\mathscr{A}_S$ is contained in some triangle of Σ_S.

Firstly we prove (1). Let C be a maximal cube containing T and let $c(\alpha), c(\sigma)$ and $c(\tau)$ be the vertices of T, where $c(\sigma)$ and $c(\tau)$ are successive in the canonical cycle of C and $c(\alpha)$ is either the center of a 2-face of C ($\dim \alpha = 1$) or the center of C itself ($\dim \alpha = 0$).

If $\dim \alpha = 1$, the definition of Σ_S yields that $C \cap S = C' \cap S$, where $C' \neq C$ is a maximal cube such that $c(\alpha)$ is the center of the common 2-face $C \cap C'$. Then, Lemma 8.5 implies that $C \cap S = \mathrm{st}_3(\alpha; R^3)$ and hence $c(\alpha) \in \mathscr{A}_S$. In case $\dim \alpha = 0$ the definition of Σ_S yields that the (equivalent) conditions of Lemma 8.3 hold, and so $c(\alpha) \in \mathscr{A}_S$. Furthermore, by Lemma 8.9 one also gets $c(\sigma \cap \tau) \in \mathscr{A}_S$ in both cases; thus, T is the union of the triangles $\langle c(\alpha), c(\sigma \cap \tau), c(\sigma) \rangle$ and $\langle c(\alpha), c(\sigma \cap \tau), c(\tau) \rangle$ of $\mathscr{A}_S$, as depicted in Fig. 8.

In order to prove (2) we observe that any vertex $c(\alpha) \in \mathscr{A}_S$ belongs to $|\Sigma_S|$. Namely, this is clear for $\dim \alpha = 3$. For $\dim \alpha = 2$ it follows from Lemma 8.10, for $\dim \alpha = 1$ from Lemma 8.4, and for $\dim \alpha = 0$ from Lemma 8.3, respectively.

Next, we check that any edge $L = \langle c(\alpha_1), c(\alpha_2) \rangle \in \mathscr{A}_S$ lies in some triangle of Σ_S.

Assume $\dim \alpha_1 = 0$. Then the elementary cube C with center $c(\alpha_1) = \alpha_1$ is maximal with respect to S by Lemma 8.3. If in addition $\dim \alpha_2 = 3$, L is actually an edge in Σ_S. In case $\dim \alpha_2 = 2$ then we derive from Lemma 8.10 that L is part of the triangle $\langle c(\alpha_1), c(\sigma), c(\tau) \rangle \in \Sigma_S$, where $\sigma, \tau \in \mathrm{cell}_3(R^3)$ are such that $\alpha_2 = \sigma \cap \tau$. Finally, if $\dim \alpha_2 = 1$, by Lemmas 8.6 and 8.4 we are under the (equivalent) conditions of Lemma 8.8; therefore, $\mathrm{st}_3(\alpha_2; S)$ consists of two successive 3-cells α_3, α_4 in the canonical cycle of C, and so $L \subseteq \langle c(\alpha_1), c(\alpha_3), c(\alpha_4) \rangle \in \Sigma_S$.

Let now assume $\dim \alpha_1 = 1$. We are under the (equivalent) conditions of either Lemma 8.7 or 8.8. In any case the result follows by the same arguments above.

Finally, in case $\dim \alpha_1 = 2$ then $\dim \alpha_3 = 3$ and the result is immediate after Lemma 8.10.

We now proceed to check that $\dim \mathscr{A}_S \leqslant 2$. Indeed, if the cells $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, with $\dim \alpha_i = i$, determine a 3-simplex in $\mathscr{A}_S$ we derive from Lemma 8.3 that the elementary cube C with center $c(\alpha_0) = \alpha_0$ is maximal with respect to S. Moreover, $c(\alpha_2) \in \mathscr{A}_S$ yields that $\alpha_3, \alpha_4 \in S$, with $\alpha_2 = \alpha_3 \cap \alpha_4$, are successive in the canonical cycle of C; see Lemma 8.10. Since $\{\alpha_3, \alpha_4\} \subseteq \mathrm{st}_3(\alpha_1; S)$ we are not under the (equivalent) conditions of Lemma 8.8, and hence the (equivalent) conditions of Lemma 8.7 hold; see Lemma 8.4. But then we are in contradiction with Lemma 8.6. Therefore $\dim \mathscr{A}_S \leqslant 2$ as it was claimed.

Finally, we will show that any triangle $\langle c(\alpha_1), c(\alpha_2), c(\alpha_3) \rangle \in \mathscr{A}_S$ is contained in some triangle of Σ_S. Indeed, the no existence of tetrahedra implies $\dim \alpha_3 = 3$, and the previous arguments for the edge $\langle c(\alpha_1), c(\alpha_2) \rangle$ show that $\langle c(\alpha_1), c(\alpha_2) \rangle \subseteq \langle c(\alpha_1), c(\sigma), c(\tau) \rangle \in \Sigma_S$, where $\alpha_2 = \sigma \cap \tau$ and σ and τ are successive 3-cells in some canonical cycle. Then, if $\dim \alpha_2 = 2$ it is immediate that $\alpha_3 \in \{\sigma, \tau\}$. Otherwise, if $\dim \alpha_2 = 1$ then $\dim \alpha_1 = 0$ and, as above, we are under the (equivalent) conditions of Lemma 8.8. Thus, $\alpha_3 \in \mathrm{st}_3(\alpha_2; S) = \{\sigma, \tau\}$ and the proof is now finished. □

Once the “if” part of Theorem 8.1 has been proved, we proceed with the proof of the “only if” part (Proposition 8.14). Next result will be used in that proof as well as to extend to the infinite case the local characterization of strong 26-surfaces (Theorem 9.1).

Proposition 8.11. *Let S be a connected f^{BM}-surface in (R^3, f^{BM}). Then S is a strongly separating object.*

Proof. According to Theorem 7.5, $\mathrm{cell}_3(R^3) - S$ has two 6-components C_1 and C_2. Moreover, these components are determined by the connected components X_1, X_2 of $|\mathscr{A}_{R^3}| - |\mathscr{A}_S| = \mathbb{R}^3 - |\mathscr{A}_S|$. Namely, $C_i = \{\sigma \in \mathrm{cell}_3(R^3) \mid c(\sigma) \in X_i\}$ (see Theorem 4.2). So it will be enough to show that for all $\sigma \in S$ both 6-components are 6-adjacent to σ. For this we use that $(B^3, B^2) = (|\mathrm{st}(c(\sigma); \mathscr{A}_{R^3})|, |\mathrm{st}(c(\sigma); \mathscr{A}_S)|)$ is a relative ball in $(\mathbb{R}^3, |\mathscr{A}_S|)$; see Proposition A.4 in the appendix. Furthermore, if D_1, D_2 are the two components of the difference $D_\sigma = B^3 - B^2$, Proposition A.4 also implies the existence of cells $\alpha_1, \alpha_2 \in R^3$ such that $c(\alpha_i) \in D_i$ $(i = 1, 2)$. Then, applying Lemma 8.13 (see below) to α_1 and α_2 we get that $\mathrm{st}_3(\alpha_i; R^3) - S$ $(i = 1, 2)$ are non-empty sets contained in different 6-components of $\mathrm{st}_3^*(\sigma; R^3) - S \subseteq \mathrm{cell}_3(R^3) - S$ which are 6-adjacent to σ. □

In the previous proof we have just used that $(|\mathrm{st}(c(\sigma); \mathscr{A}_{R^3})|, |\mathrm{st}(c(\sigma); \mathscr{A}_S)|)$ is a relative ball for each $\sigma \in S$, and furthermore the characterization of the components of the difference $D_\sigma = |\mathrm{st}(c(\sigma); \mathscr{A}_{R^3})| - |\mathrm{st}(c(\sigma); \mathscr{A}_S)|$ given in Lemma 8.13 below. To prove this result, we need the following lemma, which is an immediate consequence of Remark 7.1(2).

Lemma 8.12. *Let O be an object in (R^3, f^{BM}) and $\sigma \in O$. Let α be a face of σ with $c(\alpha) \notin \mathscr{A}_O$ (i.e., $f^{BM}(O,\alpha)=0$). Then $\mathrm{st}_3(\alpha; R^3) - O$ is a non-empty set contained in a 6-component of $\mathrm{st}_3^*(\sigma; R^3) - O$ which is 6-adjacent to σ.*

Lemma 8.13. *Let S be an f^{BM}-surface in (R^3, f^{BM}) and $\sigma \in S$. Given two faces $\alpha, \beta < \sigma$ with $c(\alpha), c(\beta) \notin \mathscr{A}_S$ the centroids $c(\alpha)$ and $c(\beta)$ are in the same component of $D_\sigma = |st(c(\sigma); \mathscr{A}_{R^3})| - |st(c(\sigma); \mathscr{A}_S)|$ if and only if $st_3(\alpha; R^3) \cup st_3(\beta; R^3) - S$ is contained in a 6-component of $st_3^*(\sigma; R^3) - S$ which is 6-adjacent to σ.*

Proof. Assume that $c(\alpha)$ and $c(\beta)$ lie in the same component of D_σ, and let $(c(\alpha_i))_{i=0}^m$ be a simplicial path in D_σ from $c(\alpha)$ to $c(\beta)$; see Proposition 4.11. We have $\alpha_0 = \alpha$, $\alpha_m = \beta$ and either $\alpha_i < \alpha_{i-1}$ or $\alpha_{i-1} < \alpha_i$, $i = 1, \dots, m$. Since

$$\mathrm{st}_3(\alpha; R^3) \cup \mathrm{st}_3(\beta; R^3) - S \subseteq \left(\bigcup_{i=0}^{m} \mathrm{st}_3(\alpha_i; R^3) \right) - S,$$

it will suffice to show that the latter is part of a 6-component of $\mathrm{st}_3^*(\sigma; R^3) - S$.

We will proceed inductively. For $i = 0$ the result follows from Lemma 8.12. Assume then that $\bigcup_{i=0}^{k-1} \mathrm{st}_3(\alpha_i; R^3) - S \subseteq A$, where A is a 6-component of $\mathrm{st}_3^*(\sigma; R^3) - S$ which is 6-adjacent to σ. If $\alpha_{k-1} < \alpha_k$ then $\mathrm{st}_3(\alpha_k; R^3) \subseteq \mathrm{st}_3(\alpha_{k-1}; R^3)$ and the result follows for k. In case $\alpha_k < \alpha_{k-1}$ we already know that $\mathrm{st}_3(\alpha_k; R^3) - S$ is contained in a 6-component B of $\mathrm{st}_3^*(\sigma; R^3) - S$ by Lemma 8.12, and $A = B$ since $\emptyset \neq \mathrm{st}_3(\alpha_{k-1}; R^3) - S \subseteq A \cap (\mathrm{st}_3(\alpha_k; R^3) - S)$.

Conversely, given 3-cells $\sigma_\alpha, \sigma_\beta \notin S$ with $\alpha < \sigma_\alpha$ and $\beta < \sigma_\beta$ they are joined by a 6-path in $\mathrm{st}_3^*(\sigma; R^3) - S \subseteq \mathrm{cell}_3(R^3) - S$ and hence the vertices $c(\sigma_\alpha), c(\sigma_\beta) \in \mathbb{R}^3 - |\mathscr{A}_S|$ are joined by a path in $\mathbb{R}^3 - |\mathscr{A}_S|$. Therefore both $c(\alpha)$ and $c(\beta)$ lie in the same component of $\mathbb{R}^3 - |\mathscr{A}_S|$ and by the property of relative balls (see Lemma A.2 in Appendix A) they are in the same connected component of D_σ. $\square$

We are now ready to show the “only if” part of Theorem 8.1.

Proposition 8.14. *Let S be an f^{BM}-surface in (R^3, f^{BM}). Then S is a near strong 26-surface.*

Proof. To prove this result it will suffice to check that properties (1)–(4) in Definition 6.1 hold for each $\sigma \in S$.

By Proposition 8.11, S is a strongly separating object and thus $\mathrm{st}_3^*(\sigma; R^3) - S$ has at least two 6-components. Hence property (1) will follow if we show that $N_{26}(\sigma; \mathbb{Z}^3 - S) = \mathrm{st}_3^*(\sigma; R^3) - S$ has at most two 6-components which are 6-adjacent to σ. For this, let $\sigma_1, \sigma_2 \in \mathrm{st}_3^*(\sigma; R^3) - S$ be two 3-cells which are 6-adjacent to σ. We apply Lemma 8.13 to $\alpha_1 = \sigma_1 \cap \sigma$ and $\alpha_2 = \sigma_2 \cap \sigma$ to derive that both σ_1 and σ_2 belong to the same

6-component of $\mathrm{st}_3^*(\sigma;R^3)-S$ if and only if the centroids $c(\alpha_1)$ and $c(\alpha_2)$ belong to the same component of $D_\sigma=|\mathrm{st}(c(\sigma);\mathscr{A}_{R^3})|-|\mathrm{st}(c(\sigma);\mathscr{A}_S)|$. Now the result follows from Lemma A.2 since $(|\mathrm{st}(c(\sigma);\mathscr{A}_{R^3})|,|\mathrm{st}(c(\sigma);\mathscr{A}_S)|)$ is a relative ball in $(\mathbb{R}^3,|S|)$; see Proposition A.4.

Next, we prove property (2); that is, the set $G_6(\sigma;\mathbb{Z}^3-S)$ has exactly two 6-components. Furthermore, we will show that these components are $G_6(\sigma;A_1^\sigma)$ and $G_6(\sigma;A_2^\sigma)$, where A_i^σ $(i=1,2)$ are the 6-components of $\mathrm{st}_3^*(\sigma;R^3)-S$ which are 6-adjacent to σ.

Obviously each 6-component of $G_6(\sigma;\mathbb{Z}^3-S)$ is contained in either A_1^σ or A_2^σ. So, to prove property (2), it will suffice to check that $G_6(\sigma;\mathbb{Z}^3-S)$ has at most two 6-components. Let $\sigma_1,\sigma_2\in G_6(\sigma;\mathbb{Z}^3-S)$ be two 3-cells 6-adjacent to σ. Furthermore, assume that both σ_1,σ_2 belong to the same 6-component of $\mathrm{st}_3^*(\sigma;R^3)-S$. Then, our task will be accomplished if we show that σ_1,σ_2 are in the same 6-component of $G_6(\sigma;\mathbb{Z}^3-S)$. For this we proceed as follows. By Theorem 7.5 we know that the centroids $c(\sigma_1),c(\sigma_2)$ are in the same component $C\subseteq\mathbb{R}^3-|\mathscr{A}_S|$. Hence $c(\sigma\cap\sigma_1)$ and $c(\sigma\cap\sigma_2)$ belong also to C and moreover, by Lemma A.2, these centroids are in the same component of the difference $D_\sigma=|\mathrm{st}(c(\sigma);\mathscr{A}_{R^3})|-|\mathrm{st}(c(\sigma);\mathscr{A}_S)|$. Thus, Proposition 4.11 allows us to find a sequence of vertices $(c(\alpha_i))_{i=0}^m$ determined by a simplicial path in D_σ joining $c(\sigma\cap\sigma_1)$ to $c(\sigma\cap\sigma_2)$. We claim that $G_6(\sigma;\mathrm{st}_3(\alpha_i;R^3)-S)$ is a non-empty set for all $0\leqslant i\leqslant m$ and, furthermore, the set $A_m=\bigcup_{i=0}^m G_6(\sigma;\mathrm{st}_3(\alpha_i;R^3)-S)$ is contained in a 6-component of $G_6(\sigma;\mathbb{Z}^3-S)$. Then property (2) follows since $\{\sigma_1,\sigma_2\}\subseteq A_m$.

In order to prove the claim we firstly show that the set $G_6(\sigma;\mathrm{st}_3(\alpha_i;R^3)-S)$ is non-empty for all $0\leqslant i\leqslant m$. Otherwise, by Lemma 8.15 below, we have $\dim\alpha_i=1$, $\alpha_i\in\mathrm{supp}(S)$ and $c(\gamma_1^i),c(\gamma_2^i)\in\mathscr{A}_S$ where $\gamma_1^i,\gamma_2^i<\sigma$ are the two 2-cells with $\alpha=\gamma_1^i\cap\gamma_2^i$. In particular, $0<i<m$ since $\dim\sigma\cap\sigma_1=\dim\sigma\cap\sigma_2=2$. Moreover, for all $0\leqslant j\leqslant m$, $\alpha_j<\sigma$ but $c(\alpha_j)\notin\mathrm{st}(c(\sigma);\mathscr{A}_S)$ and hence α_{i-1} and α_{i+1} are the vertices of α_i. Nevertheless, the definition of f^{BM} (see Remark 7.1(2)) implies $f^{BM}(S,\alpha_{i-1})+f^{BM}(S,\alpha_{i+1})=1$ and so either $c(\alpha_{i-1})$ or $c(\alpha_{i+1})$ is a vertex of $\mathscr{A}_S$. This is a contradiction and so $G_6(\sigma;\mathrm{st}_3(\alpha_i;R^3)-S)\neq\emptyset$ for all $0\leqslant i\leqslant m$.

Next, we will proceed inductively to show that $A_m=\bigcup_{i=0}^m G_6(\sigma;\mathrm{st}_3(\alpha_i;R^3)-S)$ is contained in a 6-component of $G_6(\sigma;\mathbb{Z}^3-S)$. The set A_0 coincides with $G_6(\sigma;\mathrm{st}_3(\sigma\cap\sigma_1;R^3)-S)$ and the result follows from Lemma 8.15. Assume A_{k-1} is contained in a 6-component of $G_6(\sigma;\mathbb{Z}^3-S)$. If $\alpha_{k-1}<\alpha_k$ then $\mathrm{st}_3(\alpha_k;R^3)\subseteq\mathrm{st}_3(\alpha_{k-1};R^3)$ and so $A_k\subseteq A_{k-1}$. Otherwise, $\alpha_k<\alpha_{k-1}$ and

$$\emptyset\neq G_6(\sigma;\mathrm{st}_3(\alpha_{k-1};R^3)-S)\subseteq A_{k-1}\cap G_6(\sigma;\mathrm{st}_3(\alpha_k;R^3)-S)$$

shows that $G_6(\sigma;\mathrm{st}_3(\alpha_k;R^3)-S)$ is contained in the same 6-component as A_{k-1} and the result follows.

Finally we check that the two 6-components C_1 and C_2 of $G_6(\sigma;\mathbb{Z}^3-S)$ are exactly $G_6(\sigma;A_1^\sigma)$ and $G_6(\sigma;A_2^\sigma)$. Since both 6-components A_i^σ have elements which are 6-adjacent to σ, it follows that C_1 and C_2 cannot be contained in the same A_i^σ. Without loss of generality we can assume that $C_i \subseteq A_i^\sigma$, $i=1,2$. Then, for any $\rho \in C_i$, $\rho \in A_i^\sigma$ and so it is either 6-adjacent to σ or there exists $\rho' \in C_i$ which is 6-adjacent to ρ and σ. In any case $\rho \in G_6(\sigma; A_i^\sigma)$. Moreover, $G_6(\sigma; A_i^\sigma) \subseteq G_6(\sigma; \mathbb{Z}^3 - S) = C_1 \cup C_2$ and $C_i \subseteq A_i^\sigma$ yields $G_6(\sigma; A_i^\sigma) \subseteq C_i$.

To prove property (4) it will be enough to show that any $\tau \in N_6(\sigma; S)$ is 6-adjacent to both 6-components $G_6(\sigma; A_i^\sigma)$, since all elements in $G_6(\sigma; A_i^\sigma \cup \{\tau\}) - G_6(\sigma; A_i^\sigma)$ are 6-adjacent to τ. For this we consider the 2-cell $\gamma = \sigma \cap \tau$. It is clear that $c(\gamma) \in \mathscr{A}_S$. Moreover any centroid $c(\alpha) \in D_\gamma = |\mathrm{st}(c(\gamma); \mathscr{A}_{R^3})| - |\mathrm{st}(c(\gamma); \mathscr{A}_S)|$ satisfies $\alpha < \gamma$ and then $c(\alpha) \in D_\sigma$; otherwise $\gamma < \alpha$ implies that $\alpha \in \{\sigma, \tau\}$ and so $c(\alpha) \in \mathscr{A}_S$. Notice also that the pairs $(|\mathrm{st}(c(\gamma); \mathscr{A}_{R^3})|, |\mathrm{st}(c(\gamma); \mathscr{A}_S)|)$ and $(|\mathrm{st}(c(\sigma); \mathscr{A}_{R^3})|, |\mathrm{st}(c(\sigma); \mathscr{A}_S)|)$ are relative balls in $(\mathbb{R}^3, |\mathscr{A}_S|)$ and hence each D_γ and D_σ have two components which determine the components of $\mathbb{R}^3 - |\mathscr{A}_S|$. Moreover, by Proposition A.4 one finds cells $\alpha_1, \alpha_2 < \gamma$ with $c(\alpha_1)$ and $c(\alpha_2)$ lying in different components of D_γ and hence in different components of D_σ. Now Lemma 8.13 implies that $\mathrm{st}_3(\alpha_1; R^3) - S$ and $\mathrm{st}_3(\alpha_2; R^3) - S$ are contained in different 6-components of $\mathrm{st}_3^*(\sigma; R^3) - S$. Without loss of generality we assume $\mathrm{st}_3(\alpha_i; R^3) - S \subseteq A_i^\sigma$ and hence $G_6(\sigma; \mathrm{st}_3(\alpha_i; R^3) - S) \subseteq G_6(\sigma; A_i^\sigma)$. Two possibilities can occur for α_i. If $\dim \alpha_i = 0$ or $\dim \alpha_i = 1$ and $\alpha_i \notin \mathrm{supp}(S)$ then Remark 7.1(2) ensures that there exists a 3-cell in $G_6(\sigma; \mathrm{st}_3(\alpha_i; R^3) - S)$ which is 6-adjacent to τ. Otherwise, if $\dim \alpha_i = 1$ and $\alpha \in \mathrm{supp}(S)$ again Remark 7.1 gives us a 0-cell $\beta < \alpha_i$ with $f^{BM}(S, \beta) = 0$ and we apply the previous case to β.

Finally we check property (3) in Definition 6.1; that is, for each $\tau \in N_{26}(\sigma; S)$ we must show that τ is 26-adjacent to both A_1^σ and A_2^σ. Given $\tau \in N_{26}(\sigma; S)$, by Lemma 7.3 there exists $\alpha_\tau \leqslant \sigma \cap \tau$ such that $f^{BM}(S; \alpha_\tau) = 1$. In case $\dim \alpha_\tau = 2$ the result follows from property (4) already proved above. In any case, for the 1-simplex $\gamma = \langle c(\alpha_\tau), c(\sigma) \rangle \in \mathscr{A}_S$ we have the relative ball $(|\mathrm{st}(\gamma; \mathscr{A}_{R^3})|\, |\mathrm{st}(\gamma; \mathscr{A}_S)|)$; see Proposi tion A.4. Notice that the components of $D_\gamma = |\mathrm{st}(\gamma; \mathscr{A}_{R^3})| - |\mathrm{st}(\gamma; \mathscr{A}_S)|$ are contained in different components of $D_\sigma = |\mathrm{st}(c(\sigma); \mathscr{A}_{R^3})| - |\mathrm{st}(c(\sigma); \mathscr{A}_S)|$. Furthermore, Proposition A.4 gives us centroids $c(\alpha_1)$ and $c(\alpha_2)$ in each component of D_γ and hence of D_σ. Then, as above, Lemma 8.13 implies that $\mathrm{st}_3(\alpha_i; R^3) - S \subseteq A_i^\sigma$. To finish the proof we observe that either $\alpha_i \leqslant \alpha_\tau \leqslant \tau$ or $\alpha_\tau \leqslant \alpha_i$, and thus each $\rho \in \mathrm{st}_3(\alpha_i; R^3) - S$ is 26-adjacent to τ since $\tau \cap \rho$ contains either α_τ or α_i. $\square$

Lemma 8.15. *Let O be a digital object in (R^3, f^{BM}) and let $\sigma \in O$. If $\alpha < \sigma$ is such that $c(\alpha) \notin \mathscr{A}_O$ then $G_6(\sigma; st_3(\alpha; R^3) - O)$ is contained in a* 6*-component of $G_6(\sigma; \mathbb{Z}^3 - O) = G_6(\sigma; st_3^*(\sigma; R^3) - O)$. Moreover, $G_6(\sigma; st_3(\alpha; R^3) - O)$ is the empty set if and only if* $\dim \alpha = 1$, $\alpha \in \mathrm{supp}(O)$ *and $c(\gamma_1), c(\gamma_2) \in \mathscr{A}_O$ where $\gamma_1, \gamma_2 < \sigma$ are the two* 2*-cells with $\alpha = \gamma_1 \cap \gamma_2$.*

Proof. As $\alpha < \sigma$ we get $\mathrm{st}_3(\alpha; R^3) \subseteq \mathrm{st}_3^*(\sigma; R^3)$ and hence $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O) \subseteq G_6(\sigma; \mathrm{st}_3^*(\sigma; R^3) - O)$. Moreover, Remark 7.1(2) shows that $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O) = \mathrm{st}_3(\alpha; R^3) - O$ is a one-point set if $\dim \alpha = 2$. In case $\dim \alpha = 1$ and $\alpha \notin \mathrm{supp}(O)$ Remark 7.1(2) also shows that $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O)$ is 6-connected and contains at least two 6-adjacent 3-cells.

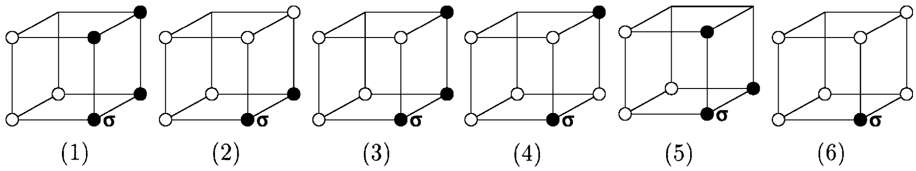


Fig. 9. Circles are elements in $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O)$ and black dots are elements in O.

The case $\dim \alpha = 1$ and $\alpha \in \mathrm{supp}(O)$ yields, by Remark 7.1(2), that $c(\beta) \notin \mathscr{A}_O$ for a vertex $\beta < \alpha$ and, since $\mathrm{st}_3(\alpha; R^3) - O \subseteq \mathrm{st}_3(\beta; R^3) - O$, we reduce this case to the case $\dim \alpha = 0$ which we proceed to analyze. Since $f^{BM}(O, \alpha) = 0$ then $\alpha \notin \mathrm{supp}(O)$ and again Remark 7.1(2) yields that a 2-face of the elementary cube C with center $c(\alpha)$ is contained in $\mathbb{Z}^3 - O$. Therefore only the six non-empty configurations depicted in Fig. 9 are possible (up to rotation or symmetry) for the set $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O)$ when $\dim \alpha = 0$.

According to the previous analysis only the case $\dim \alpha = 1$ and $\alpha \in \mathrm{supp}(O)$ allows the set $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O)$ to be empty. We next characterize this occurrence. Let $\sigma_1, \sigma_2 \in \mathrm{cell}_3(R^3)$ be the two 6-adjacent cells to σ with $\alpha = \sigma_1 \cap \sigma_2$. Then $\sigma_1, \sigma_2 \in O$, since otherwise $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O) \neq \emptyset$, and $\gamma_i = \sigma_i \cap \sigma$ $(i = 1, 2)$ satisfy the required property. Conversely, if $c(\gamma_i) \in \mathscr{A}_O$ then $\gamma_i \in \mathrm{supp}(O)$ and so $\sigma_i \in O$. Hence $G_6(\sigma; \mathrm{st}_3(\alpha; R^3) - O) = \emptyset$. $\square$

9. Final results and remarks

In this final section we show how weak lighting functions are enough to replicate the "continuous perception" associated with strong 26-surfaces. This is obtained as a corollary of the next result which extends the local characterization of finite strong 26-surfaces in [15, Theorem 6].

Theorem 9.1 (Extension of Malgouyres–Bertrand's Theorem). *Let S be a possibly infinite 26-connected object in $\mathbb{Z}^3$. Then S is a strong 26-surface if and only if it is a near strong 26-surface.*

Proof. As it was quoted in Section 6, by Theorem 6.2 and Proposition 6.3 it will suffice to show that any near strong 26-surface S is a strongly separating object. In fact this is immediate since S is an f^{BM}-surface by Proposition 8.2, and then Proposition 8.11 yields the result. $\square$

This theorem, together with the equivalence of f^{BM}-surfaces and near strong 26-surfaces proved in Theorem 8.1, shows the following characterization.

Theorem 9.2. *A connected digital object in the digital space (R^3, f^{BM}) is an f^{BM}-surface if and only if it is a strong 26-surface.*

It is worth to point out several other consequences of these results. Firstly notice that our notion of continuous analogue extends to arbitrary objects the corresponding notion defined by Malgouyres and Bertrand only for near strong 26-surfaces. This is a consequence of Proposition 8.2, where it is shown that both notions of continuous analogue coincide for near strong 26-surfaces in the digital space (R^3, f^{BM}). And, secondly, observe that a digital Jordan–Brouwer Separation Theorem for near strong 26-surfaces is obtained as a corollary of Theorems 9.2 and 7.5.

Finally, in next proposition we will show that no lighting function as defined in [3] can adequately represent all strong 26-surfaces; and so, weak lighting functions introduced in this paper are needed for this purpose. For this, we recall that lighting functions are those w.l.f.'s which satisfy the following more restrictive local property (see the paragraph before Example 3.4)
(F3) $f(O,\alpha)=f(\mathrm{st}_n(\alpha;O),\alpha)$.

Proposition 9.3. *For every weak lighting function f on R^3 satisfying $f(O,\alpha)=f(st_3(\alpha;O),\alpha)$, for any $O\subseteq cell_3(R^3)$ and $\alpha\in R^3$, there exists a strong 26-surface which is not a digital 2-manifold in (R^3,f).*

Proof. Assume, on the contrary, that there exists a digital space (R^3,f) such that any strong 26-surface is a digital 2-manifold. Thus, the continuous analogue $|\mathscr{A}_S|$ of the strong 26-surface $S=\{\sigma\in R^3 \mid c(\sigma)=(x,y,z)\in\mathbb{Z}^3, x=y\}$ is a polyhedral surface; and so, $\mathrm{lk}(c(\sigma);\mathscr{A}_S)$ is a combinatorial 1-sphere for each $\sigma\in S$. On the other hand, property (2) in Definition 3.2 implies that, for all $\sigma\in S$, the only faces $\alpha\leqslant\sigma$ that can be lighted by f for the surface S are those obtained as $\alpha=\sigma\cap\tau$, where $\tau\in S$ is 26-adjacent to σ. But all these faces must be lighted or, otherwise, $\mathrm{lk}(c(\sigma);\mathscr{A}_S)$ would not be an 1-sphere.

Now, let σ,τ be any pair of 3-cells in R^3 such that $\dim\alpha=1$, for $\alpha=\sigma\cap\tau$. A suitable translation or rotation of S yields a strong 26-surface $S_{\sigma\tau}$, with the same property obtained above for S, such that $\{\sigma,\tau\}\subseteq S_{\sigma\tau}$. Notice that $\mathrm{st}_3(\alpha;S_{\sigma\tau})=\{\sigma,\tau\}$. Then, for any digital object $O\subseteq \mathrm{cell}_3(R^3)$ such that $\mathrm{st}_3(\alpha;O)=\{\sigma,\tau\}=\mathrm{st}_3(\alpha;S_{\sigma\tau})$, property (F3) implies that $f(O,\alpha)=1$. But this leads us to a contradiction since the continuous analogue of the strong 26-surface $\bar{S}=\{\sigma\in R^3 \mid c(\sigma)=(0,y,z)\in\mathbb{Z}^3$ with $y+z$ even$\}\cup\{\sigma\in R^3 \mid c(\sigma)=(1,y,z)\in\mathbb{Z}^3$ with $y+z$ odd$\}$, depicted in Fig. 10, is not a polyhedral surface. □

We have just proved that all strong 26-surfaces cannot be simultaneously found as digital 2-manifolds in a digital space (R^3,f) such that f satisfies property (F3). Despite of this, each strong 26-surface is a digital 2-manifold in a digital space of the form (R^3,f) for a suitable lighting function f, as we show next.

Proposition 9.4. *For any strong 26-surface S there exists a w.l.f. f_S on R^3 satisfying $f(O,\alpha)=f(st_3(\alpha;O),\alpha)$, for any $\alpha\in R^3$ and $O\subseteq cell_3(R^3)$, such that S is a digital 2-manifold in (R^3,f).*

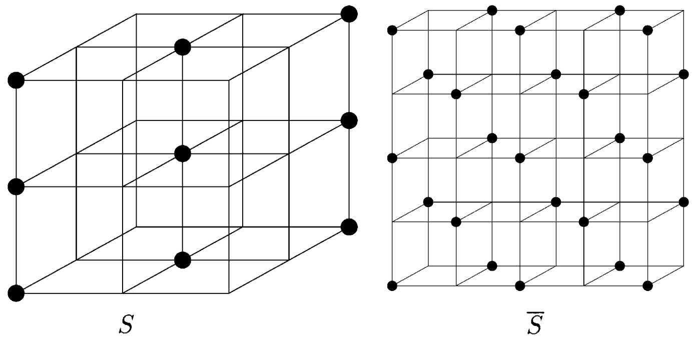


Fig. 10. A portion of the strong 26-surfaces S and $\bar{S}$ used in the proof of Proposition 9.3.

Proof. As a consequence of Theorem 6.2 and Proposition 6.3, each strong 26-surfaces is a near strong 26-surface, and so there exists a polyhedral surface Σ_S associated to S. Then, it is enough to define f_S as follows:
1. $f_S(\mathrm{cell}_3(R^3), \alpha) = 1$ for all $\alpha \in R^3$.
2. $f_S(S, \alpha) = 1$ if and only if $c(\alpha) \in |\Sigma_S|$.
3. If $S \neq O \neq \mathrm{cell}_3(R^3)$ then $f_S(O, \alpha) = 1$ if $\mathrm{st}_3(\alpha; O) = \mathrm{st}_3(\alpha; R^3)$, or $f_S(O, \alpha) = f_S(S, \alpha)$ if $\mathrm{st}_3(\alpha; O) = \mathrm{st}_3(\alpha; S)$, or $f_S(O, \alpha) = 0$ otherwise. $\square$

10. Concluding remark

Summarizing, we have generalized in this paper the notion of lighting function in order to replicate the "continuous perception" associated with (near) strong 26-surfaces. In fact, these surfaces have been characterized as digital 2-manifolds of a particular digital space (R^3, f^{BM}) defined on the grid $\mathbb{Z}^3$ by an appropriate weak lighting function. This result leads us to believe that our notion of digital surface is much broader than the corresponding notion of strong 26-surface. This belief is supported by the fact that although strong 26-surfaces generalize the Morgenthaler and Rosenfeld $(26, 6)$-surfaces (see [6, Theorem 9]) there is no such notion of strong surface generalizing $(6, 26)$-surfaces (which in some sense represent the complementary continuous perception of the former). However, we have already shown that $(6, 26)$-surfaces, actually all (α, β)-surfaces with $(\alpha, \beta) \neq (6, 6)$, can be also described as digital 2-manifolds in a suitable digital space; see [3].

Appendix A

In this appendix we recall the notion of relative ball from polyhedral topology. We use it to prove the separation properties of f^{BM}-surfaces which show that these surfaces are both near strong 26-surfaces and strongly separating objects (see Section 8).

Definition A.1. Let $M \subseteq \mathbb{R}^n$ be a polyhedral $(n-1)$-manifold. A *relative ball* in $(\mathbb{R}^n, M)$ is a pair of balls (B^n, B^{n-1}) such that $(B^{n-1}, \partial B^{n-1}) = (B^n \cap M, \partial B^n \cap B^{n-1})$.

The crucial property of relative balls is the following. The proof of this lemma is an immediate consequence of Theorem 3 in [18]; see also [2, Lemma 7].

Lemma A.2. *Any relative ball (B^n, B^{n-1}) in $(\mathbb{R}^n, M)$ verifies that $B^n - B^{n-1}$ has exactly two components each of which is contained in a distinct component of $\mathbb{R}^n - M$.*

In this way, relative balls can be used to determine the components of the complement of a polyhedral manifold, but also of a digital manifold. For this, let us consider a w.l.f. f on a device model K with $|\mathscr{A}_K| = \mathbb{R}^m$. Assume that M is a connected digital $(m-1)$-manifold in (K, f) and let (B^m, B^{m-1}) be a relative ball in $(\mathbb{R}^m, |\mathscr{A}_M|)$. Then Theorem 4.2 shows that the M-components of $\text{cell}_n(R^n) - M$ are determined by the components of $\mathbb{R}^m - |\mathscr{A}_M|$ and so Lemma A.2 implies that they are also determined by the components of $B^m - B^{m-1}$.

Next proposition gives us enough relative balls for our purposes.

Proposition A.3. *Let K be a triangulation of $\mathbb{R}^n$ and let $L \subset K$ be a full subcomplex of K such that L is a combinatorial $(n-1)$-manifold without boundary. Then for any simplex $\sigma \in L$ the pair $(B^n, B^{n-1}) = (|st(\sigma; K)|, |st(\sigma; L)|)$ is a relative ball in $(\mathbb{R}^n, |L|)$. Moreover both components of $D_\sigma = |st(\sigma; K)| - |st(\sigma; L)|$ contain vertices of K.*

Proof. Given $\sigma \in L$ with $\dim \sigma = k$, $\text{lk}(\sigma; K)$ is an $(n-k-1)$-sphere and $\text{lk}(\sigma; L)$ is an $(n-k-2)$-sphere. Moreover,

$$(\text{st}(\sigma; K), \text{st}(\sigma; L)) = (\sigma \cdot \text{lk}(\sigma; K), \sigma \cdot \text{lk}(\sigma; L)) \cong (B^n, B^{n-1}),$$

where "$\cdot$" denotes the join of two simplicial complexes. Furthermore, since L is a full subcomplex we have $\text{st}(\sigma; L) = \text{st}(\sigma; K) \cap L$ and $\text{lk}(\sigma; L) = \text{lk}(\sigma; K) \cap L$. In addition the equalities $\partial \text{st}(\sigma; K) = \partial\sigma \cdot \text{lk}(\sigma; K)$ and $\partial \text{st}(\sigma; L) = \partial\sigma \cdot \text{lk}(\sigma; L)$ are easily checked. These equalities yield $\partial \text{st}(\sigma; L) = \text{st}(\sigma; L) \cap \partial \text{st}(\sigma; K)$ and so the pair (B^n, B^{n-1}) above is a relative ball in $(\mathbb{R}^n, |L|)$.

In order to check the last part of the proposition, let C_i $(i = 1, 2)$ the components of D_σ. It is a well-known fact that the topological closure $\bar{C}_i$ of C_i coincides with the union $C_i \cup |L|$. Moreover $C_i \cup |L|$ is a polyhedron. In fact $C_i \cup |L|$ is the underlying polyhedron of the subcomplex $J_i = \{\rho \in K \mid \rho \leqslant \tau \text{ and } \tau \cap C_i \neq \emptyset\}$. Also one checks that J_i is a full subcomplex of K. Now let $x \in D_\sigma \cap C_1$ and let $\gamma \in K$ be the simplex with $x \in \mathring{\gamma}$. We claim that at least a vertex of γ lies in C_1. Otherwise $x \in \gamma \in J_2$ since J_2 is a full subcomplex. □

Finally, from Proposition A.3 one gets immediately

Proposition A.4. *Let (K, f) be a digital space with $|\mathscr{A}_K| = \mathbb{R}^m$ and let M be a digital $(m-1)$-manifold in (K, f). Then, the pair $(|st(\gamma; \mathscr{A}_K)|, |st(\gamma; \mathscr{A}_M)|)$ is a relative ball in*

$(\mathbb{R}^m, |\mathscr{A}_M|)$ for all simplex $\gamma \in \mathscr{A}_M$. Moreover, for each component $C \subseteq |st(\gamma;\mathscr{A}_K)| - |st(\gamma;\mathscr{A}_M)|$ there exists $\beta \in K$ with $c(\beta) \in C$.

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